মানুষের জ্ঞান ও ভাবকে বইয়ের মধ্যে সঞ্চিত করিবার যে একটা প্রচুর সুবিধা আছে, সে কথা কেহই অস্বীকার করিতে পারে না। কিন্তু সেই সুবিধার দ্বারা মনের স্বাভাবিক শক্তিকে একেবারে আচ্ছন্ন করিয়া ফেলিলে বুদ্ধিকে বাবু করিয়া তোলা হয়।

— *রবীন্দ্রনাথ ঠাকুর*

ভারতের একটা mission আছে, একটা গৌরবময় ভবিষ্যৎ আছে, সেই ভবিষ্যৎ ভারতের উত্তরাধিকারী আমরাই। নূতন ভারতের মুক্তির ইতিহাস আমরাই রচনা করছি এবং করব। এই বিশ্বাস আছে বলেই আমরা সব দুঃখ কষ্ট সহ্য করতে পারি, অন্ধকারময় বর্তমানকে অগ্রাহ্য করতে পারি, বাস্তবের নিষ্ঠুর সত্যগুলি আদর্শের কঠিন আঘাতে ধূলিসাৎ করতে পারি।

— *সুভাষচন্দ্র বসু*

Any system of education which ignores Indian conditions, requirements, history and sociology is too unscientific to commend itself to any rational support.

— *Subhas Chandra Bose*

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Choice Based Credit System (CBCS)

UG
SELF LEARNING MATERIAL

Skill Enhancement Course (SEC)

Mathematics (HMT)
Logic and Sets
SE - MT - 11

PREFACE

In a bid to standardise higher education in the country, the University Grants Commission (UGC) has introduced Choice Based Credit System (CBCS) based on five types of courses viz. *core, discipline specific, generic elective, ability* and *skill enhancement* for graduate students of all programmes at Honours level. This brings in the semester pattern, which finds efficacy in sync with credit system, credit transfer, comprehensive continuous assessments and a graded pattern of evaluation. The objective is to offer learners ample flexibility to choose from a wide gamut of courses, as also to provide them lateral mobility between various educational institutions in the country where they can carry acquired credits. I am happy to note that the University has been accredited by NAAC with grade 'A'.

UGC (Open and Distance Learning Programmes and Online Learning Programmes) Regulations, 2020 have mandated compliance with CBCS for U.G. programmes for all the HEIs in this mode. Welcoming this paradigm shift in higher education, Netaji Subhas Open University (NSOU) has resolved to adopt CBCS from the academic session 2021-22 at the Under Graduate Degree Programme level. The present syllabus, framed in the spirit of syllabi recommended by UGC, lays due stress on all aspects envisaged in the curricular framework of the apex body on higher education. It will be imparted to learners over the *six* semesters of the Programme.

Self Learning Materials (SLMs) are the mainstay of Student Support Services (SSS) of an Open University. From a logistic point of view, NSOU has embarked upon CBCS presently with SLMs in English / Bengali. Eventually, the English version SLMs will be translated into Bengali too, for the benefit of learners. As always, all of our teaching faculties contributed in this process. In addition to this we have also requisitioned the services of best academics in each domain in preparation of the new SLMs. I am sure they will be of commendable academic support. We look forward to proactive feedback from all stakeholders who will participate in the teaching-learning based on these study materials. It has been a very challenging task well executed, and I congratulate all concerned in the preparation of these SLMs.

I wish the venture a grand success.

Professor (Dr.) Subha Sankar Sarkar
Vice-Chancellor

Netaji Subhas Open University

Under Graduate Degree Programme

Choice Based Credit System (CBCS)

Subject : Honours of Mathematics (HMT)

Course : Logic and Sets
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Netaji Subhas Open University

Logic and Sets
(HMT)

Course : Logic and Sets
Course Code : SE-MT-11

Unit-1 ❑ Basics of Logic and Truthtables

Structure

1.0 Objectives

After going through this unit the learners should be able to :

- Understand the definitions of proposition, truth table, logical connectives, tautology, contradiction.
- Represent logical expressions by using symbols, variables and connectives in place of natural languages such as English to remove vaguness.

1.1 Introduction

Mathematics is based on logic and logic is the study of reasoning or argumentation. It is therefore necessary to know the rules of logic to distinguish between valid and invalid arguments. Let us discuss it with an example.

Suppose you walk past a barber's shop one day and see a poster in the shop that says—

"Do you shave yourself ? If not, come in and I will shave you ! I shave everyone who does not shave himself and no one else."

This seems fairly simple. But if a question occurs as "who shaves the barber ?", then no matter if we try to answer this question we get into trouble.

Now if we say that the barber shaves himself then we get into trouble, because the barber shaves only those who do not shave themselves. So if he shaves himself then he does not shave himself which is self-contradictory.

Again if we say that the barber does not shave himself then also we get into trouble. Because the barber shaves everyone who does not shave himself. Thus if he does not shave himself then he must shave himself which again a self-contradictory.

Even if we try to avoid the truble by a tricky answer such as the barber is a woman then also it will be absurd because the woman either shaves herself or does not shave herself. It she shaves herself then she is one of the people who is not shaved by the barber.

Again if she does not shave herself then she is one of the people who is shaved by the barber. Both the cases are impossible.

Thus the barber can neither shave himself nor not shave himself. Hence the question "who shaves the barber ?" is not answerable. This means that 'barber' does not exist. There is no entity that satisfies the description of the barber.

The above paradox is known as barber's paradox, discovered by British mathematician and philospher Bertrand Russell, at the begining of the twentieth century.

Thus it is necessary to know the mathematical logic i.e. the part of mathematics concerned with the formal language, formal reasoning, the nature of mathematical proof and other aspects of the foundation of mathematics.

Now we start with a brief overview of mathematical logic. Next we review some basic notations from elementary set theory which provides a medium or communicating mathematics in a precise and clear way.

1.2 Propositions

A proposition (or statement) is a declarative sentence which is either true or false but not both. A declarative sentence declares or states a fact. Thus imperative, optative, interrogative or exclamatory sentences are not propositions. The truth or falsity of a statement or proposition is called its truth value. If a statement is true then we denote its truth value by 'T' (or by '1'). Again if a statement is false then we denote its truth value by 'F' (or by '0'). Thus the working nature of a statement or proposition can be compared to natural working switch in a circuit,

so statement → True ('T' or '1')
→ False ('F' or '0')

and switch → on ('1')
→ off ('0')

We shall use the lower case letters of the alphabets (such as a, b or c) to represent a statement or proposition. As for example

a : $3 + 4 = 7$

b : Kolkata is the capital of U.S.A.

and c : $n = 3$ is a solution of $n<2$

are propositions. Since the statement a is true i.e. $3 + 4 = 7$ is true hence the truth value of a is T (or 1). But the statements b and c are false hence the truth value of b or c if F (or 0). On the otherhand the following sentences are not propositions or statements.

(i) What a beautiful scenery it is !

(ii) Will you go to the college tomorrow ?

(iii) Get up in morning.

This is due to the fact that these are not declarative sentences. Again the following sentence

"I am lying"

is not a proposition because it is neither true nor false i.e. it has no truth value. Actually this sentence is a self contradictory or paradox known as liar's paradox in logic. Because if the speaker is indeed lying then the liar is telling the truth but a liar can not tell the truth. Again if the speaker is a truth-teller then a truth teller can not tell that he is lying.

A statement which is true is known as valid statement and a statement which is false is known as invalid statement. A statement is called simple or primitive if it cannot be broken down into simpler proposition or statement. Thus the truth value of a simple or primitive statement does not explicitly depend on any other statement. A statement is said to be compound if it is formed by two or more simple statements by various logical connectives such as 'and', 'or', 'not', 'if then' etc. We shall discuss about the various logical connectives subsequently. Thus in the preceding statements a, b or c are primitive statements.

Again the statement

p : 3 + 4 = 7 or n = 3 is a solution of $n < 2$ is a compund statement because it is formed by two primitive statements a and b by a logical connective 'or'. Simple statements which when combined to form a compound statement are called substatements or subpropositions. Thus in the compund statement p, a and b are subpropositions.

1.3 Truth Table

A table that shows the relationship between the truth value of a compound statement S (p, q, r ....) and the truth values of its substatements p, q, r.... etc. is called the truth table of statement S. We know that truth value of a compound proposition S (p, q, r, ...) depends exclusively upon the truth values of its variables i.e. the truth values of its subpropositions p, q, r .... etc. Thus for preparing truth table we take 1st, 2nd, 3rd ..... etc. columns for the variables p, q, r .... etc. and we have to take enough rows in the table to allow these variables. Thus for single variable p, $2^1 = 2$ rows are necessary, for two variables p and q, $2^2 = 4$ rows are necessary, for three variables p, q and r, $2^3 = 8$ rows are required. As for example if S be compound statement of three variables p, q, r then 1st three columns and eight rows can be taken as follow:

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

The eight possible truth assignments for p, q, r can be listed in any order. Thus we can also write eight rows as follows (by writting '0' for false and '1' for true)

p	q	r
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

1.4 Logical connectives, Elementary operations on statements

Two or more connectives are combined to form a compound statement by using symbols. These symbols are called logical connectives. Logical connectives are given below :

Words	Symbols
and	$\wedge$
or	$\vee$
implies that (if ....then)	$\rightarrow$
implies and is implied by (if and only if)	$\leftrightarrow$
exclusive or	$\veebar$

1.4.1 Negation :

Negation is an unary operation on a statement. If p is a statement then its negation is denoted by $\sim p$ or $\neg p$. Negation of p or $\sim p$ is an another statement formed by writing "it is false that" before p or if possible by inserting the word "not" in p. We read negation of p i.e. $\sim p$ as "not p". By definition truth table for negation is as follow :

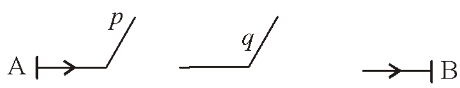
p	$\sim p$
T	F
F	T

1.4.2 Conjunction :

Conjunction is a binary operation on statements. Two statements p and q can be combined by the word "and" to form a compound proposition called the conjunction of their original propositions. Symbolically we denote it as $p \wedge q$ and and read as "p and q". The truth table for conjunction is :

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

It is clear from the above table that $p \wedge q$ takes the value T when and only when both p and q are true. As for example if p : Amit is a student, q : Amit lives in Kolkata then $p \wedge q$ is equivalent to Amit is a student and he lives in Kolkata. Above conjunction can also be explained through circuits where the two switches p, q say are connected in the series. Current will pass only from A to B if both p and q are of 'on' state.



1.4.3 Disjunction :

Disjunction is also a binary operation on statements. Any two propositions can be combined by the word "or" to form a compound proposition called disjunction of the original propositions. Symbolically we denote it as $p \vee q$ and read as "p or q". The truth table for disjunction is :

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

From the above table it is clear that $p \vee q$ takes the value F when and only when both p and q are false. The word "or" is used in above in inclusive sense. In the inclusive sense "p or q" means "p or q or both". Thus by our symbolic language "$p \vee q$" we always mean "p and / or q". As for example if

p : Train left early

q : The watch of Mita is going slow

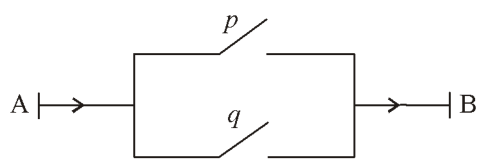
then $p \wedge q$ is equivalent to :

Train left early or the watch of Mita is going slow.

If we went to combine two propositions p and q by the word "or" in exclusive sense which is known as "xor" then we denote the compoun proposition as $p \veebar q$. In the exclusive sense of "or", "p xor q" we mean "p or q but not both". This means the compound statement $p \veebar q$ is true if either p or q is true but not true if both of the statements are true. The truth table for $p \veebar q$ is :

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

The disjunction $p \vee q$ can also be explained through circuits where the two switches p, q say are connected in parallel. Current will pass from A to B when either or both the switches are of 'on'-state.



1.4.4 Implication (or Conditional) :

Implication (or Conditional) is also a binary operation. Any two statements p and q can be combined by using the word "if" before p and the word "then" before q. Thus the implication is "if p then q". Symbolically we denote it as $p \rightarrow q$ where p is called the hypothesis of implication and q is called the conclusion. Alternatively we can also say that "p is sufficient for q" or "q is necessary for p". The truth table for $p \rightarrow q$ is :

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The table shows that $p \rightarrow q$ is false when and only when p is true but q is false.

As for example if p : Bus reaches in time, q : Bimal will attend the college then $p \rightarrow q$ is equivalent to : If bus reaches in time then Bimal will attend the college. It means that if bus does not reach in time then Bimal will or will not attend the college. But it guaranted that Bimal will attend the college provided bus reaches in time.

1.4.5 Biimplication (or Biconditional) :

The binary operation 'biconditional' or 'biimplication' combines two statements p and q to form the new statement "p if and only if q" denoted by $p \leftrightarrow q$. Alternatively we can say that "p is necessary and sufficient for q". Its truth tabl is :

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

The table says that $p \leftrightarrow q$ is true when and only when both p and q have the same kind of truth values. As for example if

p : The unit digit of an integer is 0 or 5

q : The number is divisible by 5

then $p \leftrightarrow q$ is equivalent to : The unit digit of an integer is 0 or 5 if and only if it is divisible by 5.

1.5 Tautology, Contradiction

A compound statement is called a tautology if in its truth table, the column under that compound statement contains only T's i.e. if its truth function takes always the value T. As for example "p or not p" that is $p \vee \sim p$ is a tautology. This can be verified from the following truth table.

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Similarly a compound statement is called a contradiction if in its truth table the column under that compound statement contains only F's i.e. if its truth function takes always the value F. As for example the proposition "p and not p" that is $p \wedge \sim p$ is a contradiction.

This can be verified by looking the folowing

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Thus the negation of a tautology is a contradiction and the negation of a contradiction is a tautology.

If a compound statement is neither tautology nor contradiction then the compound statement is called a contingency.

1.6 Examples

1.6.1 Determine whether each the following sentences is a propostion. Also identify the primitive statements.

(i) Switch off the fan.

(ii) Smoking is injurious to health.

(iii) If 9 is greater than 6 then 10 is greater than 7.

(iv) x is an integer.

(v) The sum of 2 and 3 is greater than 6.

Solution :

(i) It is an imperative sentence hence it is not a statement or proposition.

(ii) It is a declarative sentence, hence it is a proposition.

(iii) It is also declarative sentence hence it is a proposition.

(iv) Here we do not know the sentence is true or false because we do not know the value of x. Hence it is not a proposition.

(v) It is a proposition because we know that the given sentence is true or false.

Since $2 + 3 \nmid 6$ hence the truth value of the statement or proposition is 'F' (or False). However it is a proposition or statement.

2nd Part :

The statements (ii) and (v) are primitive or simple statements since these statements cannot be broken into two or more sentences. But the statement (iii) is a compound statement since it is formed by two simple statements "9 is greater than 6" and "10 is greater than 7" by the word "if ... then"

1.6.2 Let *p, q, r, s* denote the following statements

p : Kolkata is in U.P.

q : $5 + 2 > 8$.

r : Delhi is the capital of India.

s : 54 is divisible by 9.

Find the truth values of

(i) $p, q, r, s.$

(ii) $\sim p, \sim r$

(iii) $p \vee r, p \vee q, q \vee s$

(iv) $p \wedge q, p \wedge r, \sim p \wedge \sim q, r \wedge s$

(v) $p \veebar q, p \veebar r, r \veebar s$

(vi) $p \rightarrow r, p \rightarrow q, r \rightarrow q, s \rightarrow r$

(vii) $p \leftrightarrow q, q \leftrightarrow r, r \leftrightarrow s$

Solution :

(i) Since Kolkata is not in U.P. hence the truth value of p is F.

Again since $5 + 2 = 7 \nmid 8$ hence the turth value of q is F.

Again since it is true that Delhi is the capital of India hence the truth value of r is T.

Similarly since 54 is divisible by 9 hence the truth value of s is T.

(ii) Since the truth value of p is F hence the truth value of "not p" i.e. $\sim p$ is T. Similarly the truth value $\sim r$ is F.

(iii) $p \vee r$ will be true if and only if either p or r or both are true. As r is true hence the truth value of $p \vee r$ is T. Similarly the truth value of $p \vee q$ is F since neither p nor q is true. The truth value of $q \vee s$ is T as s is true.

(iv) We know the $p \wedge q$ is true if and only if both p and q are true. But here p is false hence the truth value of $p \wedge q$ is F. By the same argument $p \wedge r$ is false hence its truth value is F. Again as $\sim p$ is true and $\sim q$ is true hence $\sim p \wedge \sim q$ is also true and hence its truth value is T. Similarly the truth value of $r \wedge s$ is T.

(v) We know the $p \veebar q$ is true if and only if exactly one statement p(or q) is true and the remaing statement q (or p) is false. Now since p is false and q is false hence $p \veebar q$ is false and hence the truth value of $p \veebar q$ is F. Again since p is false and r is true hence the truth value of $p \veebar r$ is T. Again since both r and s are true hence the truth value of $r \veebar s$ is F.

(vi) We know that $p \rightarrow r$ is false if and only if p is true and r is false. In our problem p is false hence whatever be the truth value of r; $p \rightarrow r$ is true i.e. its truth value is T. Similarly the truth value of $p \rightarrow q$ is T. Again since r is true and q is false hence the truth value of $r \rightarrow q$ is F. Again since s is true and r is also true hence the truth value of $s \rightarrow r$ is T.

(vii) We know that $p \leftrightarrow q$ is true if and only if both p and q have the same truth value. In our problem since p and q have the same truth value F hence the truth value of $p \leftrightarrow q$ is T. Again since q is false and r is true i.e. q and r have not the same truth value hence $q \leftrightarrow r$ is false i.e. the truth value of $q \leftrightarrow r$ is F. Again since both r and s are true hence the truth value of $r \leftrightarrow s$ is T.

1.6.3 Let *p, q, r* denote the following statements about a quadrilateral ABCD.

p : ABCD is a rectangle.

q : ABCD is a square.

r : ABCD is a parallelogram.

Translate each of the following into an English sentence.

(i) $\sim p$ (ii) $p \vee q$ (iii) $q \to p$ (iv) $p \wedge \sim q$ (v) $r \to p$ (vi) $p \leftrightarrow r$

Solution :

(i) $\sim p$: ABCD is not a rectangle.

(ii) $p \vee q$: ABCD is a rectangle or a square.

(iii) $q \to p$: If ABCD is a square, then it is a rectangle.

(iv) $p \wedge \sim q$: ABCD is a rectangle but it is not a square.

(v) $r \to p$: If ABCD is a parallelogram then it is a rectangle.

(vi) $p \leftrightarrow r$: ABCD is a rectangle if and only if it is a parallelogram.

1.6.4 Find the truth table of

(i) $\sim p \wedge q$

(ii) $(p \wedge q) \to p$

(iii) $\sim (p \vee \sim q) \to p$

(iv) $(p \to q) \to r$

where p, q, r denote primitive statements. Also identify the compound statement which is a tautology.

Solution :

(i) Truth table of $\sim p \wedge q$

p	q	$\sim p$	$\sim p \wedge q$
T	T	F	F
T	F	F	F
F	T	T	T
F	F	T	F

(ii) Truth table of $(p \wedge q) \rightarrow p$

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

(ii) Truth table of $\sim (p \vee \sim q) \rightarrow p$

p	q	$\sim p$	$p \vee \sim q$	$\sim (p \vee \sim q)$	$\sim (p \vee \sim q) \rightarrow p$
T	T	F	T	F	T
T	F	T	T	F	T
F	T	F	F	T	F
F	F	T	T	F	T

(iv) Truth table of $(p \rightarrow q) \rightarrow r$

p	q	r	$p \rightarrow q$	$(p \rightarrow q) \rightarrow r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

2nd Part :

In the truth table of $(p \wedge q) \rightarrow p$ we have seen that $(p \wedge q) \rightarrow p$ contains only T in the last column. This means $(p \wedge q) \rightarrow p$ is true for all truth values of p and q. Hence $(p \wedge q) \rightarrow p$ is a tautology.

1.7 Converse, contrapositive and inverse propositions of an implication

Given an implication statement of the form "if p then q" we can create three related statements :

We know that a conditional statement consists of two parts, the hypothesis in the "if" clause and a conclusion in the "then" clause. For example in the statement "If it rains then Hafija cancels dancing". Here "It rains" is the hypothesis. "Hafija concels dancing" is the conclusion.

To form the converse of a conditional statement, interchange the hypothesis and the conclusion. Thus the converse of "If it rains Hafija cancels dancing" is "If Hafija cancels dancing then it rains".

To form the inverse of a conditional statement take the negation of both the hypothesis and conclusion. Thus the inverse of "If it rains then Hafija cancels dancing" is "If it does no rain then Hafija does not cancel dancing".

To form the contrapositive of a conditional statement, interchange the hypothesis and the conclusion of the inverse statement. Thus the contrapositive of "If it rains then Hafija cancels dancing" is "If Hafija does not cancel dancing then it does not rain".

Thus converse, inverse and contrapositive statements of the conditional statement $p \to q$ (i.e. "If p then q") are $q \to p$ (i.e. "If q then p"), $\sim p \to \sim q$ (i.e. "If not p then not q") and $\sim q \to \sim p$ (i.e. "If not q then not q") respectively.

Let us construct the truth tables for conditional proposition $p \to q$ and the converse, inverse and contrapositive statements of the conditional proposition $p \to q$.

p	q	Conditional $p \to q$	Converse $q \to p$	$\sim p$	$\sim q$	Inverse $\sim p \to \sim q$	Contrapositive $\sim q \to \sim p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

1.7.1 Examples :

Write the converse, inverse and contrapositive of each of the following implications. For each implication, determine its truth value as well as the truth values of its corresponding converse, inverse and contrapositive.

(i) If $2 + 3 = 5$ then $\cos(2\pi) + \cos(3\pi) = \cos(5\pi)$

(ii) If $2 > 1$ and $5 > 3$ then $7 > 6$

Solution :

(i) We know that $2 + 3 = 5$ is true. Again $\cos(2\pi) + \cos(3\pi) = 1 - 1 = 0$ but $\cos(5\pi) = -1$

Thus $\cos(2\pi) + \cos(3\pi) = \cos(5\pi)$ is false. Hence the implication :

If $2 + 3 = 5$ then $\cos(2\pi) + \cos(3\pi) = \cos(5\pi)$ is false. This is due to the fact that in the above implication the hypothesis is true but the conclusion is false.

Again the converse statement of the implication is :

If $\cos(2\pi) + \cos(3\pi) = \cos(5\pi)$ then $2 + 3 = 5$. The above statement is true since its hypothesis $\cos(2\pi) + \cos(3\pi) = \cos(5\pi)$ is false.

Again the inverse statement of the given implication is :

If $2 + 3 \neq 5$ then $\cos(2\pi) + \cos(3\pi) \neq \cos(5\pi)$. This statement is true since its hypothesis is false.

Again the contrapositive statement of the given implication is :

If $\cos(2\pi) + \cos(3\pi) \neq \cos(5\pi)$ then $2 + 3 \neq 5$.

The above statement is false since in the above proposition the hypothesis is true but its conclusion is false.

(ii) The given implication is $p \rightarrow q$ where $p : 2 > 1$ and $5 > 3$ and $q : 7 > 6$

Here p is ture and q is also true. Hence $p \rightarrow q$ is true.

This mean the implication, "If $2 > 1$ and $5 > 3$ then $7 > 6$" is true.

Converse of the above implication is : If $7 > 6$ then $2 > 1$ and $5 > 3$.

This is also true as both p and q are true.

Inverse of given implication is : If $2 \ngtr 1$ or $5 \ngtr 3$ then $7 \ngtr 6$

The above statement is also true since the hypothesis of the above implication is false.

Again contrapositive statement of the given implication is : If $7 \ngtr 6$ then $2 \ngtr 1$ or $5 \ngtr 3$

It is also true since the hypothesis of the statement is false.

1.8 Exercise

1.8.1 Chose the correct option :

(i) Which of the following is not a statement.

(a) 2 is the only even prime number.

(b) 72 is not divisible by 3.

(c) Come here.

(d) Ram and Shyam are friends.

(ii) The conditional statement of "You will get nice gift after the dinner" is

(a) If you take the dinner then you will get a nice gift.

(b) You get a nice gift if and only if you take the dinner.

(c) You take the dinner and get a nice gift.

(d) None of the above.

(iii) Which of the following is the inverse of the proposition "If a number is a prime then it is odd"

(a) If a number is not a prime then it is odd.

(b) If a number is not a prime then it is not odd.

(c) If a number is not odd then it is not a prime.

(d) If a number is not odd then it is a prime.

(iv) Let p and q be two statements. Then $p \vee q$ is false if

(a) p is false and q is true.

(b) both p and q are false.

(c) both p and q are true.

(d) p is true and q is false.

(v) If p : A quadrilateral is a parallelogram, q : The opposite sides are parallel. Then the statement "A quadrilateral is a parallelogram if and only if the opposite sides are parallel" is represented as

(a) $p \vee q$ (b) $p \rightarrow q$ (c) $p \wedge q$ (d) $p \leftrightarrow q$

(vi) Which of the following is true for the statements p and q ?

(a) $p \vee q$ is true when atleast one of p and q is true.

(b) $p \to q$ is true when p is true and q is false.

(c) $p \leftrightarrow q$ is true only when both p and q are false.

(d) $\sim (p \vee q)$ is true only when both p and q are false.

(vii) The converse of the contrapositive of the conditional $p \to \sim q$ is

(a) $p \to q$ (b) $\sim p \to \sim q$ (c) $\sim q \to p$ (d) $\sim p \to q$

(viii) Which of the following propositions is tautology ?

(a) $(p \vee q) \to p$ (b) $p \vee (q \to p)$ (c) $p \vee (p \to q)$ (d) $p \to (q \to p)$

1.8.2 Using the following statements :

p : Temperature is above 40°C.

q : Rita will go for movie.

Write the following statements in symbolic forms :

(i) Temperature is above 40°C but Rita will not go for movie.

(ii) If temperature is above 40°C then Rita will go for move.

(iii) Rita will go for movie if and only if temperature is not above 40°C.

1.8.3 Find the truth table for — (i) $p \vee \sim q$ (ii) $\sim p \wedge \sim q$

1.8.4 Verify that the proposition $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

1.8.5 Prove that the following are tautologies.

(i) $(p \wedge (p \to q)) \to q$

(ii) $(p \to q) \leftrightarrow (\sim p \vee q)$

1.8.6 If p and q are true, determine whether the statement $(p \vee q) \wedge [(\sim p) \vee (\sim q)]$ is true or false.

1.8.7 How many rows are needed for the truth value of the compound statement $(\sim r \wedge (p \vee q)) \to (\sim p \wedge s)$ where p, q, r and s are all primitive statements ?

1.9 Answers to the exercise 1.8

1.8.1 (i) c (ii) a (iii) b (iv) b (v) d (vi) a, d (vii) d (viii) c

1.8.2 (i) $p \wedge \sim q$ (ii) $p \to q$ (iii) $q \leftrightarrow \sim p$

1.8.3 (i)

p	q	$\sim q$	$p \vee \sim q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

(ii)

p	q	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

1.8.4

p	q	$p \wedge q$	$p \vee q$	$\sim (p \vee q)$	$(p \wedge q) \wedge \sim (p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

From the last column it follows that $(p \wedge q) \wedge \sim (p \vee q)$ is a contradiction.

1.8.5 (i)

p	q	$p \to q$	$p \wedge (p \to q)$	$(p \wedge (p \to q)) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

From the last column we see that for all possible values of p and q the given proposition is true. Hence it is a tautology.

(ii)

p	q	$\sim p$	$p \to q$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

From the last column it follows that the given proposition is a tautology.

1.8.6 False

1.8.7 $2^4 = 16$

1.10 Summary

In this unit we have learnt how to translate statement into symbolic form and how to construct a truth table. Also the structure of conditional statement has been discussed. Learner can now distinguish a compound statement by tautology, contradiction or contingency. It is expected that learner should understand thoroughly the usage of the elementary operations on statements.

Unit-2 ❑ Logical Equivalences and the use of Quantifiers

Structure

2.0 Objectives

After going through this unit the learner should be able to :

- Determine the logical equivalence of two statements.
- Know the meaning of predicates and quantifiers.

- Understand the mathematical logic that is closely connected with the foundation of mathematical analysis and theoretical computer science.

2.1 Introduction

In Mathematics, sometimes we wish to know when the entities we are studying are equal or essentially the same. As for example for real x, we have $x^2 < 4$ if $-2 < x < 2$ and cpmversely $-2 < x < 2$ it $x^2 < 4$. Thus for real x, the two statements $x^2 < 4$ and $-2 < x < 2$ are esentially the same. So we need to define the logical equivalence of propositions.

2.2 Propositional equivalence : Logical equivalences

2.2.1 Logical equivalence of propositions :

Let $P(p, q, ...)$ denote an expression constructed from propositions $p, q, ...$ which can take on the value True (T) or False (F) and the logical connectives $\wedge$, $\vee$ and $\sim$ (and others which are already discussed). Such an expression $P(p, q, ...)$ is also a proposition and its truth value depends exclusively upon the truth values of its variables i.e. the truth values of propositions $p, q, ...$ Two propositions (or statements) $P(p, q, ...)$ and $Q(p, q, ...)$ are said to be logically equivalent or simply equivalent, denoted by $P(p, q,......) \equiv Q(p, q,.....)$ or $P(p, q,.....) \Leftrightarrow Q(p, q,......)$ when the statement $P(p, q,.....)$ is true or false if and only if the statement $Q(p, q,......)$ is true or false respectively. Thus $P(p, q,.....)$ and $Q(p, q,......)$ have the same truth tables because $P(p, q,.......)$ and $Q(p, q,......)$ have the same truth values for their primitive components. As for example we can write $\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$ or $\sim (p \wedge q) \equiv \sim p \vee \sim q$, since both the propositions $\sim (p \wedge q)$ and $\sim p \vee \sim q$ have the identical truth tables as follows :

p	q	$p \wedge q$	$\sim (p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

p	q	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

Let us consider an another example. Let us find the truth tables of $p \rightarrow q$ and $\sim p \vee q$ as follows :-

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

p	q	$\sim p$	$\sim p \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

We observe that the truth values $p \rightarrow q$ are exactly same as the truth values of $\sim p \vee q$.

Hence we can writ $p \rightarrow q \Leftrightarrow \sim p \vee q$.

Let p and q be the primitive statements as

p : Bus reaches in time.

q : Bimal will attend the college.

Then $p \rightarrow q$ is the statement : If bus reaches in time then Bimal will attend the college.

We have seen that $p \rightarrow q \Leftrightarrow \sim p \vee q$.

Again $\sim p \vee q$ is the stateemnt : Bus does not reach in time or Bimal will attend the college.

Hence the two statements "If bus reaches in time then Bimal will attend the college" and "Bus does not reach in time or Bimal will attend the college" are logically equivalent.

2.2.2 Algebra of propositions (The Laws of Logic) :

Using the concepts of logical equivalence, tautology, contradiction we state the following laws of propositons known as laws for algebra of propositions :

For any primitive statements p, q, r, any tautology T_0 and any contradiction F_0.

(i) Law of double negation (or Involution law) :

$\sim \sim p \Leftrightarrow q$

(ii) Idempotent laws :

$p \vee p \Leftrightarrow p, \quad p \wedge p \Leftrightarrow p$

(iii) Associative laws :

$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r), (p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

(iv) Commutative laws :

$p \vee q \Leftrightarrow q \vee p, p \wedge q \Leftrightarrow q \wedge p$

(v) Distributive laws :

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r), p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(vi) Complement laws (or Inverse laws) :

$p \vee \sim p \Leftrightarrow T_0, p \wedge \sim p \Leftrightarrow F_0$

(vii) Identity laws :

$p \vee F_0 \Leftrightarrow p, p \wedge T_0 \Leftrightarrow p$

(viii) Domination laws :

$p \vee T_0 \Leftrightarrow T_0, p \wedge F_0 \Leftrightarrow F_0$

(ix) Absorption laws :

$p \vee (p \wedge q) \Leftrightarrow p, p \wedge (p \vee q) \Leftrightarrow p$

(x) De Morgan's laws :

$\sim (p \vee q) \Leftrightarrow \sim p \wedge \sim q, \sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$

We can verify the above laws by constructing truth tables for the statements of each side of '$\Leftrightarrow$' sign as we did in 2.1 to prove $\sim (p \wedge q) \Leftrightarrow \sim p \vee \sim q$ or $(p \to q) \Leftrightarrow \sim p \vee q$.

One interesting point can be noted here. From the above mentioned pair of laws from (ii) to (x) we observe that from the first law of each pair, second law can be deduced by interchanging the connectives $\wedge$ and $\vee$ and by interchanging T_0 and F_0 and vice-versa that is first law can also be deduced from second law by interchanging $\wedge$ and $\vee$ and by interchanging T_0 and F_0.

This principle is called the Principle of Duality.

In general for any two statements s and t if $s \Leftrightarrow t$ then

dual of the statement $s \Leftrightarrow$ dual of the statement t.

Again we have seen earlier that for any statement p and q. $(p \to q) \Leftrightarrow \sim p \vee q$

This means $(p \to q) \leftrightarrow (\sim p \vee q)$ is a tautology.

Now in the above property we can replace p or q or both by any other compound statement.

As for example if we replace p by compound statement $(s \vee t)$ we can write as $(s \vee t) \to q \Leftrightarrow \sim (s \vee t) \vee q$ i.e. $((s \vee t) \to q) \leftrightarrow (\sim (s \vee t) \vee q)$ will also be a tautology. This rule is called substitution rule.

2.3 Examples

2.3.1 Prove that $\sim (p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \sim q \Leftrightarrow \sim p \leftrightarrow q$

Solution :

Let us construct the truth tables for $\sim (p \leftrightarrow q)$, $p \leftrightarrow \sim q$, $\sim p \leftrightarrow q$ separately

p	q	$p \leftrightarrow q$	$\sim (p \leftrightarrow q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	T	F

p	q	$\sim q$	$p \leftrightarrow \sim q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	F

p	q	$\sim p$	$\sim p \leftrightarrow q$
T	T	F	F
T	F	F	T
F	T	T	T
F	F	T	F

From the above truth tables we observe that the last column of each table are identical.

Hence $\sim (p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \sim q \Leftrightarrow \sim p \leftrightarrow q$.

2.3.2 Write down the negation of each statement as simple as possible.

(i) If John is rich then he is happy.

(b) If it is cold then Sita will not go to school.

(c) Sum of two integers is odd if and only if one integer is odd and other integer is even.

Solution :

(i) Let p : John is rich

q : He is happy.

Then the given statement is $p \rightarrow q$

Hence its negation is $\sim (p \rightarrow q)$

We know that $p \rightarrow q \Leftrightarrow \sim p \vee q$

$\therefore$ By De Morgan's law

$\sim (p \rightarrow q) \Leftrightarrow \sim (\sim p \vee q) \Leftrightarrow \sim \sim p \wedge \sim q \Leftrightarrow p \wedge \sim q$

Hence the simplified form of negation of the given stateemnt is John is rich and he is not happy.

(ii) Let p : It is cold.

q : Sita will go to school.

Then the given statement is $p \to \sim q$

Hence the negation of the statement is $\sim (p \to \sim q)$

$\Leftrightarrow \sim (\sim p \vee \sim q) \Leftrightarrow p \wedge q$ by De Morgan's law.

Hence the negation of the given statement is :

It is cold and Sita will go to school.

(iii) Let p : Sum of two integers is odd.

q : One integer is odd.

r : The other integer is even.

Then the given statement is $p \leftrightarrow (q \wedge r)$

Hence its negation is

$\sim (p \leftrightarrow (q \wedge r)) \Leftrightarrow \sim p \leftrightarrow (q \wedge r)$

Hence the negation of the statement in simplified form is
Sum of two integers is not odd
if and only if one integer is odd
and the other integer is even.

2.3.3 If $(p \wedge \sim r) \to (q \vee r)$ is false then what will be the truth value of p ?

Solution :

Let us prepare a truth table for $(p \wedge \sim r) \to (q \vee r)$ as follows :

p	q	r	$p \wedge \sim r$	$q \vee r$	$(p \wedge \sim r) \to (q \vee r)$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	F	T	T
F	F	F	F	F	T

We observe from the last column and fourth row that $(p \wedge \sim r) \to (q \vee r)$ is false only when the truth value of p is T. Hence p must be true.

2.3.4 Using the laws of logic prove that $\sim (p \vee q) \vee (\sim p \wedge q) \Leftrightarrow \sim p$

Solution : $\sim (p \vee q) \vee (\sim p \wedge q)$

$\Leftrightarrow (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ by De Mogran's law

$\Leftrightarrow \sim p \wedge (\sim q \vee q)$ by distribution law

$\Leftrightarrow \sim p \wedge T_0$ by complement law

$\Leftrightarrow \sim p$ by identity law

$\therefore \sim (p \vee q) \vee (\sim p \wedge q) \Leftrightarrow \sim p$ ***proved.***

2.3.5 Choose the correct option.

(i) If $S(p, q, r) = (\sim p) \vee \sim (q \wedge r)$ is a compound statement then $S(\sim p, \sim q, \sim r)$ is

(a) $\sim S(p, q, r)$ (b) $S(p, q, r)$ (c) $p \vee (q \wedge r)$ (d) $p \vee (q \vee r)$

(ii) Consider the following statements

p : Sarama is brilliant

q : Sarama is rich

r : Sarama is honest

The negation of the statement

"Sarama is brilliant and dishonest if and only if Sarama is rich."

can be equivalenty expressed as

(a) $\sim q \leftrightarrow \sim p \vee r$ (b) $\sim q \leftrightarrow \sim p \wedge r$ (c) $\sim q \leftrightarrow p \vee \sim r$

(d) $\sim q \leftrightarrow p \wedge \sim r$

(iii) The statement $p \rightarrow (q \rightarrow p)$ is equivalent to

(a) $p \rightarrow q$ (b) $p \rightarrow (p \vee q)$ (c) $p \rightarrow (p \rightarrow q)$ (d) $p \rightarrow (p \wedge q)$

Solutions :

(i) $S(p, q, r) = (\sim p) \vee \sim (q \wedge r)$

$\therefore S(\sim p, \sim q, \sim r) = (\sim \sim p) \vee \sim (\sim q \wedge \sim r) = p \vee (q \vee r)$

Hence the correct option is (d)

(ii) The given statement is

$(p \wedge \sim r) \leftrightarrow q$

Hence negation of the statement is

$\sim ((p \wedge \sim r) \leftrightarrow q) \Leftrightarrow \sim q \leftrightarrow (p \wedge \sim r)$

Hence the correct option is (d)

(iii) $p \to (q \to p) \Leftrightarrow \sim p \vee (q \to p) \Leftrightarrow (\sim p) \vee (\sim q \vee p)$

$\Leftrightarrow (\sim p) \vee (p \vee \sim q)$ by commutative law

$\Leftrightarrow (\sim p \vee p) \vee (\sim q)$ by associative law

$\Leftrightarrow T_0 \vee (\sim q)$ by complement law

$\Leftrightarrow T_0$ by domination law

Again $p \to (p \vee q) \Leftrightarrow (\sim p) \vee (p \vee q)$

$\Leftrightarrow (\sim p \Leftrightarrow p) \Leftrightarrow q$ by associative law

$\Leftrightarrow T_0 \Leftrightarrow q$ by complement law

$\Leftrightarrow T_0$ by domination law

Hence $p \to (q \to p) \Leftrightarrow p \to (p \vee q)$

Hence the correct option is (b)

2.3.6 "If p and q are arbitrary statements such that $p \to q$ is a tautology then we say that p logically implies q and we write $p \Rightarrow q$."

Using the above definition prove that $p \Rightarrow (p \vee q)$

Solution :

The truth table of $p \to (p \vee q)$ is as follows :

p	q	$p \vee q$	$p \to p \vee q$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

Since $p \to (p \vee q)$ is a tautology, hence $p \Rightarrow (p \vee q)$

2.4 Predicates and quantifiers

2.4.1 Definition of predicate and quantifier :

Consider the sentence "n is less than 5". In this sentence without knowing the value of n we do not know the given sentence is true or false. So it is not a statement. But when certain value is given to the variable n then this sentence becomes a statement. This type of sentence is called an open statement. Moreover the open statement "n is less than 5" has two parts. The first part the variable n is the subject of the open staement and the second part "is less than 5" is the predicate. It refers to a property that the open statement can have. The open statement "n is less than 5" can be denoted by $p(n)$ where p denotes the predicate "is less than 5" and n is the variable.

Thus a predicate or a propositional functin or an open statement defined on a set A (say) is an expression $p(x)$ which has the property that $p(a)$ is true or false for each $a \in A$.

That is $p(x)$ becomes a statement (with a true value) whenever any element a is substituted for the variable x. The set A is called the domain of $p(x)$ or the universe or universe of discourse for the open statement and the set T_p of all such elements $a \in A$ for which $p(a)$ is true is called the truth set of $p(x)$.

In general an open statement may have more than one variable. As for example we can denote the open statement "$x = y + 2$" as $p(x, y)$

Obviously $p(1, 1)$ is not true since $1 \neq 1 + 2$

but $p(3, 1)$ is true since $3 = 1 + 2$.

Quantifiers are words, expressions or phrases that indicate the no. of elements that an open statement pertains to. Let $p(x)$ be a propositional function or open statement defined on a set A. Then $p(x)$ could be true for all $x \in A$, or for some $x \in A$ or for no $x \in A$. As for example if $p(x)$ be defined on set N (the set of natural no.) as "$x + 1 > 4$" then its truth set is $T_p = \{x \in N/x + 1 > 4\} = \{4, 5, 6, ...\}$ consisting of all integers greater than 3.

Again if $p(x)$ be "$x + 1 < 1$" defined on the set N then its truth set is null set as for all $x \in N$ $x + 1 \nless 1$.

More over if $p(x)$ be "$x + 1 > 1$" defined on N then its truth set is N since for all $x \in N$ $x + 1 > 1$.

In mathematical logic there are two quantifiers "there exist" and "for all". The first one is called existential quantifer and the second one is called universal quantifier.

<u>Existential Quantifier :</u>

Let $p(x)$ be propositional function defined on a set A. Consider the expression

$$(\exists x \in A)p(x) \text{ or } \exists x, p(x)$$

which reads "There exists an $x \in A$ such that $p(x)$ is a true statement" or simply "for some x, $p(x)$". The symbol "$\exists$" which read "there exists" or "for some" or for "at least one" is called the existential quantifier.

Thus $(\exists\ x \in A)\ p(x)$ is equivalent to $T_p = \{x / x \in A, p(x)\} \neq \emptyset$.

$\emptyset$ being empty set.

As for example the proposition $(\exists\ n \in N)(n + 1 < 4)$ is true

since $\{n \in N / n + 1 < 4\} = \{1, 2\} \neq \emptyset$

But the proposition $(\exists\ n \in N)(n + 4 < 1)$ is false

since $\{n \in N/n + 4 < 1\} = \emptyset$ (the empty set)

We note that the expression $p(x)$ by itself is an open statement and therefore has no truth value but $(\exists\ x \in A)\ p(x)$ i.e. $p(x)$ preceded by $\exists$, always has a truth value.

Universal Quantifiers :

Let $p(x)$ be propositional function defined on a set A. Consider the expression

$$(\forall x \in A) p(x) \text{ or } \forall x\ p(x)$$

which reads "For every x in A, $p(x)$ is a true statement" or simply "For all x, $p(x)$". The symbol '$\forall$' which reads "for all" or "for every" is called universal quantifier. The statement $(\forall\ x \in A)\ p(x)$ is equivalent to the statement $T_p = \{x : x \in A,\ p(x)\} = A$ that is the truth set of $p(x)$ is the entire set A.

We note that the expression $p(x)$ by itself is an open statement or condition and therefore has no truth value. However $(\forall\ x \in A)\ p(x)$ that is $p(x)$ preceded by a quantifier $\forall$, does have a truth value.

As for example the proposition $(\forall\ n \in N)(n + 2 > 1)$ is true

since $\{n \in N : n + 2 > 1\} = \{1, 2, 3 \ldots\ldots\} = N$

But the proposition $(\forall\ n \in N)(n + 1 > 2)$ is false

since $\{n \in N : n + 1 > 2\} = \{2, 3, 4 \ldots\} \neq N$

2.4.2 Binding variables :

The variable x in each open statement $p(x)$ is called a free variable. As x varies over the universe for an open statement the truth value of the statement may vary. In contrast to the open statement $p(x)$, the statement $\exists\ x, p(x)$ has a fixed truth value.

In symbolic representation $\exists\ x, p(x)$, the variable x is said to be a binding variable or bound variabl as it is bound by the existential quantifier $\exists$.

Thus a variable whose occurance is bounded by a quantifier is called a binding variable or bound variable. Variables not bound by any quantifiers are called free variables. As for example in the expression $(\exists\ x \in N)(x + 1 > 10)$ the variable x is a bound variable since a quantifier '$\exists$' is used on variable x.

Again the expression $(\exists\, x \in N)(x + 1 > y)$ the variable y is a free variable since no quantifier is used on the variable y, but the variable x is bound since a quantifier '$\exists$' used on the variable x.

2.4.3 Order of quantifiers :

Consider the propositional function $(\forall\, x)(\exists\, y)\; p(x, y)$, where $p(x, y)$ means "x loves y". This means for every individual x there exists at least one individual y such that x loves y. This means everybody loves some body. But on the other hand if we write $(\exists\, y)(\forall\, x)\; p(x, y)$ then it means there exists some individual y who is loved by everyone. This is obviously not the same meaning as previous.

So we have to careful about the order of quantifiers and order to be followed from left to right.

2.4.4 Negations of quantified statements :

Consider the statement : "All real numbers are complex numbers". We know its negation as "It is not the case that all real numbers are complex numbers" or equivalently we can say that "There exist at least one real number which is not a complex number". Symbolically if R be the set of all real numbers then "All real numbers are complex numbers" can be writen as $(\forall\, x \in R)$(x is complex) and its negation i.e. $\sim (\forall\, x \in R)$(x is complex) is equivalent to $(\exists\, x \in R)$(x is not complex)

Again if we denote "x is complex" as $p(x)$

then we have $\sim (\forall\, x \in R)\; p(x) \Leftrightarrow \exists\, x\, (\sim p(x))$

In a similar manner the rule for negation of a proposition which contains the existential quantifier is as follows :

$\sim (\exists\, x\; p(x)) \Leftrightarrow \forall\, x (\sim p(x))$

Thus the negation of the statement "There exists a prime number greater than 1000" is "Every prime number is less or equal to 1000"

We know that if p is a statement its negation is denoted by $\sim p$ and $\sim p$ means the statement "not p". In a similar manner if $p(x)$ is a propositional function then $\sim p(x)$ has the meaning as "The statement $\sim p(a)$ is true when $p(a)$ is false and vice versa"

Similarly for two propositional function $p(x)$ and $q(x)$

$p(x) \wedge q(x)$ means

"The statement $p(a) \wedge q(a)$ is true when $p(a)$ and $q(a)$ are true".

More over $p(x) \vee q(x)$ means

"The statement $p(a) \vee q(a)$ is true when $p(a)$ or $q(a)$ is true".

We can also verify the laws of propositions also hold for propositional functions. As for example De Morgan's law for two propositional functions $p(x)$ and $q(x)$ will be

$$\sim (p(x) \wedge q(x)) \Leftrightarrow (\sim p(x)) \vee (\sim q(x))$$

$$\text{and } \sim (p(x) \vee q(x)) \Leftrightarrow (\sim p(x)) \wedge (\sim q(x))$$

2.4.5 Propositional functions with more than one variable :

A propositional function of two variables defined over a product set $L = A \times B$ is an expression $p(x, y)$ where $x \in A$ and $y \in B$ which has the property that $p(a, b)$ is true or false for any $(a, b) \in L$.

Such a propositional function $p(x, y)$ has no truth value. But if a propositional function preceded by a quantifier for each variable then the propositional function will be a statement and has a truth value.

As for example let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5\}$ and let $p(x, y)$ denote "$x + y = 6$" Then $p(x, y)$ is a propositional function defined on $L = A \times B$. Now $p(x, y)$ is not a proposition, since we do not know whether $x + y = 6$ is true or false. But if write $\forall\, x\ \exists\, y,\ p(x, y)$ that is "For every x there exists y such that $x + y = 6$" then it is a statement and its truth value is T, since if $x = 1$, $y = 5$ or if $x = 2$, $y = 4$ and so on.

Again if we write $\exists\, y\ \forall\, x,\ p(x, y)$

that is "There exist y for every x we have $x + y = 6$"

then it is also a statement and its truth values is F since no such y exists.

Thus a different ordering of the quantifires may yield a different statement which we have discussed in 2.3.4.

In a similar manner we can define a propositional function of n variables $(n \geq 3)$.

2.4.6 Negation of quantified statements with more than one variable :

Quantified statements with more than one variable may be negated by successively applying the two equivalence properties discussed in 2.3.5. Thus we have to pass $\sim$ symbol through the statement from left to right by changing each $\forall$ to $\exists$ and each $\exists$ to $\forall$.

As for example

$$\sim[\forall x \exists y,\ p(x,y)] \Leftrightarrow \exists x \sim [\exists y, p(x,y)] \Leftrightarrow \exists x \forall y, \sim p(x,y)$$

As for example let us consider the example in 2.3.6 where $\forall x \exists y, p(x, y)$ means "For every x, there exists y such that $x + y = 6$"

Then its negation will be

$$\sim[\forall x \exists y,\ p(x,y)] \Leftrightarrow \exists x \forall y, \sim p(x,y)$$

that is "There exists x such that for all y, $x + y \neq 6$".

2.5 Examples

2.5.1 Let $p(x)$, $q(x)$ denote the following open statements :

$p(x)$: $x^2 = 3x - 2$

$q(x)$: $2x < 7$

$r(x)$: $x + 1$ is even

If N, the set of natural no. is the universe, what are the truth value of the following statements ?

i) $p(2)$ ii) $\sim q(3)$ iii) $p(3) \vee r(5)$ iv) $p(1) \wedge (q(2) \vee r(4))$ v) $p(1) \rightarrow q(4)$

vi) $q(4) \rightarrow p(1)$ vii) $\exists x\, p(x)$ viii) $\forall x\, p(x)$ ix) $q(3) \leftrightarrow r(3)$ x) $r(4) \leftrightarrow q(4)$

Solutions :

(i) $p(2)$ is true since $2^2 = 3.2 - 2$

(ii) Since $2.3 < 7$ is true hence $q(3)$ is true and hence $\sim q(3)$ is false.

(iii) $p(3)$ is false since $3^2 \neq 3.3 - 2$ and $r(5)$ is true since $5 + 1$ is even.

Hence $p(3) \vee r(5)$ is true.

(iv) $p(1)$ is true since $1^2 = 3.1\text{-}2$, $q(2)$ is true since $2.2 < 7$, $r(4)$ is false since $4 + 1$ is not even.

Hence $q(2) \vee r(4)$ is true and hence $p(1) \wedge (q(2) \vee r(4))$ is true.

(v) $p(1)$ is true and $q(4)$ is false since $2.4 \nless 7$ hence $p(1) \rightarrow q(4)$ is false.

(vi) $q(4)$ is false and $p(1)$ is true hence $q(4) \rightarrow p(1)$ is true.

(vii) Since for $x = 1$, $p(x)$ is true hence $\exists x\, p(x)$ is true.

(viii) Since for $x = 3$, $p(x)$ is false hence $\forall\, x\, p(x)$ is false.

(ix) Since $q(3)$ and $r(3)$ both have the same truth value T hence $q(3) \leftrightarrow r(3)$ is true.

(x) Since $r(4)$ and $q(4)$ both have the same truth value F hence $r(4) \leftrightarrow q(4)$ is true.

2.5.2 Let $A = \{1, 2, 3, 4\}$. consider the following sentences. If it is a statement determine its truth value. If it is a propositional function, determine its truth set. Also identify the bound variable(s) and free variable(s) in each case.

(i) $(\forall x \in A)(\exists y \in A)(x + y < 6)$

(ii) $(\forall x \in A)(x^2 + y^2 < 25)$

(iii) $(\forall x \in A)(\forall y \in A)(x^2 + y^2 < 35)$

(iv) $(\forall x \in A)(\forall y \in A)(x^2 + y^2 < 25)$

(v) $(\exists y \in A)(x + y < 6)$

(vi) $(x + y < 6)$

Solutions :

(i) The open sentence is of two variables and each variable is preceded by a quantifier hence it is a statement and hence each variable is a bound variable. More over the statement is true since for $x = 1, 2, 3, 4$, we have at least one value of $y = 1$(say) such that $x + y < 6$.

(ii) The open sentence is of two variables x and y. Here only one variable x is preceded by a quantifier, hence x is a bound variable and y is a free variable. Thus it is a propositional function. We note that for every $x \in A$, $x^2 + y^2 < 25$ if and only if $4^2 + y^2 < 25$.

Hence the truth set of $y = \{y \in A / 16 + y^2 < 25\} = \{1, 2\}$.

(iii) It is also a statement since both the variables x and y are preceded by quantifiers. Thus x and y are bound variables.

Again the statement is true since for all $x, y \in \{1, 2, 3, 4\}$, $x^2 + y^2 < 35$

(iv) It is also a statement and x, y are bound variables. But the statement is false since if $x = 4$ and $y = 3$, $x^2 + y^2 < 25$ is not true.

(v) The given open statement has two variables out of which y is preceded by a quantifier. Hence it is a propositional function. Again for any $x \in \{1, 2, 3, 4\}$ we can find at least one $y \in \{1, 2, 3, 4\}$ such that $x + y < 6$ hence the truth set of $x = \{1, 2, 3, 4\}$ i.e. A itself.

Again here x is a free variable and y is a bound variable.

(vi) The given open statement has two variables without any quantifier. Hence these variable x, y are free variables. Hence it is a propositional function of x and y. Now the truth set of (x, y) is as follows :

$\{(x, y) \in A \times A / x + y < 6\}$

$= \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)\}$

2.5.3 For the universe of all integers, consider the following open statements

$p(x)$: x is even

$q(x)$: x is prime

$r(x)$: x is a perfect square.

Write down the following statements in symbolic form. Also write down negation of each statement in symbolic form and in English sentence.

(i) There exists an even integer which is prime.

(ii) If any integer is even then it is a perfect square.

(iii) If any integer is even and perfect square then it is not prime.

Also determine the truth value of the given statement and its negation. Provide counterexample for each false statement.

Solutions :

(i) $\exists x\, p(x) \wedge q(x)$

It's negation is

$\sim (\exists\, x\, p(x) \wedge q(x))$

$\Leftrightarrow \forall\, x \sim (p(x) \wedge q(x)) \Leftrightarrow \forall\, x (\sim p(x) \vee \sim q(x))$

This means "For all integer x, x is not even or x is not prime".

(ii) $(\forall x)(p(x) \rightarrow r(x))$

It's negation is

$\sim (\forall\, x\, p(x) \rightarrow r(x))$

$\Leftrightarrow \exists x (\sim (p(x) \rightarrow r(x))) \Leftrightarrow \exists x (\sim (\sim p(x) \vee r(x))) \Leftrightarrow \exists x (p(x) \wedge \sim r(x))$

This means "There exists an integer x such that it is even and not perfect square".

(iii) The given statement in symbolic form is $\forall x((p(x) \wedge r(x)) \rightarrow \sim q(x))$

The negation of the above statement is

$\exists x \sim (p(x) \wedge r(x)) \rightarrow \sim q(x)$

$\Leftrightarrow \exists x \sim (\sim (p(x) \wedge r(x)) \wedge \sim q(x)) \Leftrightarrow \exists x (p(x) \wedge r(x)) \wedge q(x)$

This mean "There exists an integer x such that it is even and perfect square and prime.

2nd Part :

(i) The given statement $\exists x(p(x) \wedge q(x))$ is true since 2 is an even integer and prime. Thus its negation

$\forall x(\sim p(x) \vee \sim q(x))$ is false and $x = 2$ is the counter example.

(ii) The given statement $\forall x(p(x) \rightarrow r(x))$ is false. $x = 10$ is a counter example since 10 is even but not perfect square. Hence its negation $\exists x(p(x) \wedge \sim r(x))$ is true.

(iii) The given statement $\forall x((p(x) \wedge r(x)) \rightarrow \sim q(x))$ is true since every even perfect square integer is not a prime integer. Hence its negation $\exists x(p(x) \wedge r(x)) \wedge q(x)$ is false. This means there is no integer x s.t. it is even, perfect square and prime.

2.6 Exercise

2.6.1 Chose the correct option

(i) $(p \vee p) \vee (p \rightarrow (q \vee q))$ is equivalent to

a) $p \rightarrow q$ b) $p \vee q$ c) $p \wedge q$ d) $q \rightarrow p$

(ii) The proposition $p \rightarrow [q \rightarrow (p \wedge q)]$ is

a) a tautology b) a contradiction c) logicaly equivalent to $p \rightarrow q$

d) none of these.

(iii) $(p \vee q) \wedge \sim (\sim p \wedge q)$ is logically equivalent to

a) $p \rightarrow q$ b) p c) q d) $\sim p$

(iv) For any primitive statements p, q, r, the statement $p \rightarrow (q \rightarrow r)$ can be written equivalenty with exactly one occurance of the connective $\rightarrow$ is

a) $p \vee (q \rightarrow r)$ b) $(p \rightarrow q) \vee r$ c) $\sim (p \rightarrow q) \vee r$ d) $(q \rightarrow r) \vee \sim p$

2.6.2 Use truth tables to verify the logical equivalence of the following :

$(p \rightarrow (q \wedge r)) \Leftrightarrow (\sim r \rightarrow (p \rightarrow q))$

Also find the dual of the above logical equivalence.

2.6.3 Prove that the implication $p \rightarrow q$ is logically equivalent to its contrapositive statement.

2.6.4 Use substitution rule to show that $p \rightarrow (q \wedge r) \Leftrightarrow (p \wedge \sim q) \rightarrow r$.

2.6.5 Let N be the set of natural numbers. Determine the truth value of each statement

(i) $(\exists x \in N)(\exists y \in N)[(2x + y = 5) \wedge (x - y = -2)]$

(ii) $(\exists x \in N)(\exists y \in N)[(x + y = 4) \wedge (x - y = 1)]$

(iii) $(\exists x \in N)(\forall y \in N)(xy = y)$

(iv) $(\forall x \in N)(\forall y \in N)(x + 2y > y)$

(v) $(\exists y \in N)(\forall x \in N)(x.y = y)$

2.6.6 Consider the definition that the sequence $\{a_n\}_{n=1}^{\infty}$ is a null sequence is as follows :

$(\forall \in > 0)(\exists m \in N), (\forall n > m)(|a_n| < \in)$

where N is the set of natural numbers. Determine the negation of the above statement that is when the sequence $\{a_n\}_{n=1}^{\infty}$ is said to be a non-null sequence.

2.6.7 Consider the quantified statement "Every science student of H.S. has at least one course where the teacher is a laboratory assistant."

Write down the statement in symbolic form. Determine its negation in symbolic form and in an English sentence.

2.7 Answer to the exercise 2.6

2.6.1 (i) c (ii) a (iii) b (iv) d

2.6.2 Since $p \to (q \vee r) \Leftrightarrow \sim p \vee (q \vee r)$
and $\sim r \to (p \to q) \Leftrightarrow r \vee (p \to q) \Leftrightarrow r \vee (\sim p \vee q)$
Dual of the given logical equivalence is $\sim p \wedge (q \wedge r) \Leftrightarrow r \wedge (\sim p \wedge q)$

2.6.5 (i) True (ii) False (iii) True (iv) True (v) False

2.6.6 Definition that the sequence $\{a_n\}$ is a non-null sequence is as follows :

$$(\exists \in > 0)(\forall m \in N)(\exists n > m)(|a_n| \geq \in)$$

2.6.7 Let A be the set of science students of H.S. and B be the set of courses and p be the statement "The teacher is a laboratory assistant".

The given statement in symbolic form $(\forall x \in A)(\exists y \in B)(p)$

and its negation is $(\exists x \in A)(\forall y \in B)(\sim p)$

This means "There is a science student of H.S. such that in every course the teacher is not a laboratory assistant".

2.8 Summary

In this unit we have discussed about the use of truth-tables to identify logically equivalent statements. Learner can now construct combined truth-table of propositions. Two different quantifiers existential and universal have been discussed with different scenarios. The learners need to understand these topics completely for making rules of inference and decision making.

Unit-3 ❑ Set operations and the Laws of Set Theory

Structure

3.0 Objectives

After going through this unit the learner should be able to :

- Understand the fundamental concepts of sets and operations on sets like union, intersection & complementation.
- Know the laws of set theory, venn diagrams and counting principle.
- Develop mathematical theories formally by having the collections we want to the talk about as mathematical objects on their own accord.

3.1 Introduction

The concept of sets is used for foundation of various topics in mathematics. To learn sets we often talk about the collection of objects such as set of vowels, set of real numbers, a list of fruits, a bunch of keys, a group of students etc.. Here we shall adopt the naive theory of sets as developed by German mathematician George Cantor.

3.2 Sets, subsets, universal set, empty set, equality of two sets

3.2.1 Sets :

According to the definition of Cantor, a set is a well defined collection fo distinct objects of our perception or of our thought, to be conceived as a whole. The word 'well defined' refers to a speific property which makes it easy to identify whether a given object belongs to the set or not. The word 'distinct' means that objects of a set must be all different. The objects of a set are called the elements or members of the set. A set is usually denoted by capital A, B, C .... etc. and the elements are denoted by small letters *a, b, c,* ...... etc. If A is a set and '*a*' is an element of the set A we write $a \in A$ read as '*a* belongs to A'. Again if A is any set and '*a*' is not an element of A then we write $a \notin A$ read as '*a* does not belong to A'.

There are three ways to represent a set.

(i) **Tabular form :** Listing all the elements of a set seperated by comma and enclosed within curly brackets '{}'. As for example $A = \{2, 3, 7, 10\}$, $B = \{a, e, i\}$ etc.

(ii) **Descriptive from :** Describing all the elements of a set by words. As for example C = set of first ten natural numbers.

(iii) **Set Builder form :** Writing in symbolic form the common characteristic property shared by all the elements of a set. Thus if p be defining property of the element of a set S then S is expressed as

$S = \{x : x$ has the property $p\}$ or $S = \{x / x$ has the property $p\}$

This means S is the collection of all those elements x such that x has the property p. As for example $X = \{x : x$ is an even integer and $x > 10\}$.

Some sets will occur very often in the text and so we use special symbols for them. Unless otherwise specified we will let

N : the set of all natural numbers

Z : the set of all integers

E : the set of all even integers

Q : the set of rational numbers

Q^* : the set of all nonzero rational numbers

Q^+ : the set of all positive rational numbers.

R : the set of real numbers.

R^* : the set of all nonzero real numbers.

R^+ : the set of all positive real numbers.

C : the set of all complex numbers.

3.2.2 Universal set and empty set :

In any application of the theory of sets the members of all sets under investigation usually belong to some fixed large set called the univerasl set and this set is generally denoted by U or S.

For example we can take $U = \{x \in N : 1 \leq x \leq 10\}$ as a universal set if we discuss involving the sets $B = \{1, 2, 4, 5\}$, $C = \{4, 5, 10\}$, $D = \{2, 5, 8\}$ since all the elements of B, C and D are the elements of U.

A set is said to be empty set or void set (or null set) if it has no element. Such a set is doneted by ϕ. For example the set $\{x : x$ is an integer such that $x^2 - 3 = 0\}$ is an empty set since there is no integer x such that $x^2 - 3 = 0$. Thus $\{x \in Z : x^2 - 3 = 0\} = \phi$

We remember that a null set or empty set is denoted by ϕ but the set $\{\phi\}$ deontes the nonempty set whose only one elements is ϕ.

An empty set has the following properties :

(i) It is subset of any set.

(ii) Its only subset is the empty set itself.

(iii) Its number of elements is zero.

3.2.3 Subset :

If A and B are two sets such that every element of A is also an element of B then we say that 'A is a subset of B' or 'A is contained in B' or 'B contains A'. This relationship is written as $A \subseteq B$ or $B \supseteq A$. If in addition B contains an element that is not in A then we say that A is a proper subset of B and this is denoted by $A \subset B$ or $B \supset A$. Again if atleast one element of A does not belong to B then we write $A \not\subseteq B$ or $B \not\supseteq A$.

We note that for all sets A, B from a universal set U

For example $A = \{1,3,7,8,9\}$, $B = \{2,7,9\}$, $C = \{7,9\}$ then

$C \subseteq A$ and $C \subseteq B$ but $B \not\subseteq A$ since $2 \in B$ but $2 \notin A$. Furthermore since the elements of A, B, C must also belong to the universal set U hence U must at least contain the elements of the set $\{1, 2, 3, 7, 8, 9\}$.

3.2.4 Equality of two sets :

Two sets A and B are said to be equal if $A \subseteq B$ and $B \subseteq A$ and we write $A = B$. From the definition it is clear that neither order nor repetition is relevant for a general set. For example $\{2, 5, 9\} = \{5, 2, 9\} = \{9, 9, 5, 2, 9\}$

3.3 Set operations and the laws of set theory

3.3.1 Union and Intersection :

Let U be the universal set and A, B be the subsets of U. Then the union of two set A and B denoted by $A \cup B$ is the set of all eleemnts which belong to A or to B. That is $A \cup B = \{x : x \in A \vee x \in B\}$. Here 'or' is used in inclusive sense.

Again the intersection or meet of two sets A and B denoted by $A \cap B = \{x : x \in A \wedge x \in B\}$

If $A \cap B = \phi$ that is if A and B do not have any element in common then A and B are said to be disjoint or non-intersecting.

For example if $U = \{2, 3, 4, 5, 6\}$, $A = \{2, 3, 5\}$, $B = \{4, 5, 6\}$, $C = \{6\}$

then $A \cup B = \{2, 3, 4, 5, 6\}$, $A \cap B = \{5\}$, $A \cup C = \{2, 3, 5, 6\}$, $A \cap C = \phi$

3.3.2 Complement of a set :

We know that all sets under consideration at a particular time are subsets of fixed universal set U. The complement of a set $A \subseteq U$, denoted by A^c or $\overline{A}$ is the set of all elements which belong to U but which do not belong to A.

This is $\overline{A} = \{x : x \in U \wedge x \notin A\}$

For example if $U = \{2, 3, 4, 5, 6\}$ and $A = \{2, 3, 5\}$ then $\overline{A} = \{4, 6\}$.

3.3.3 Laws of Set Theory :

Sets under the operation of union, intersection and complement satisfy the various laws or identities. These laws are known as laws of the algebra of sets or simply laws of set theory. Let us enlist some of the major laws as follows :

For any sets A, B, C taken from a universal set S we have

(i) $\overline{\overline{A}} = A$ law of double complement or Involution law

(ii) $A \cup \phi = A, A \cap S = A$ Identity laws

(iii) $A \cup A = A, \ A \cap A = A$ Idempotent laws

(iv) $A \cup B = B \cup A, \ A \cap B = B \cap A$ Commutative laws

(v) $(A \cup B) \cup C = A \cup (B \cup C), (A \cap B) \cap C = A \cap (B \cap C)$ Associative laws

(vi) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Distributive laws

(vii) $A \cup (A \cap B) = A, \ A \cap (A \cup B) = A$ Absorption laws

(viii) $A \cup \overline{A} = S, \ A \cap \overline{A} = \phi$ Inverse laws

(ix) $A \cup S = S, \ A \cap \phi = \phi$ Domination laws

(x) $\overline{A \cup B} = \overline{A} \cap \overline{B}, \overline{A \cap B} = \overline{A} \cup \overline{B}$ De Morgan's Laws

All the above properties or laws can be proved by using the definition of equality which is $A = B \Leftrightarrow A \subseteq B \wedge B \subseteq A$

For exampl let us prove second equality of (v), second equality of (vii) and 1st equality of (x) and leave the rest as an exercise.

Proof of $(A \cap B) \cap C = A \cap (B \cap C)$

Proof : Let $x \in (A \cap B) \cap C \Rightarrow x \in A \cap B$ and $x \in C$

$\Rightarrow x \in A$ and $x \in B$ and $x \in C$ $\Rightarrow x \in A$ and $x \in B \cap C \Rightarrow x \in A \cap (B \cap C)$

Hence $(A \cap B) \cap C \subseteq A \cap (B \cap C)$ ....(1)

Again let $y \in A \cap (B \cap C) \Rightarrow y \in A$ and $y \in (B \cap C)$

$\Rightarrow y \in A$ and $y \in B$ and $y \in C \Rightarrow y \in A \cap B$ and $y \in C \Rightarrow y \in (A \cap B) \cap C$

Hence $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ ....(2)

Combining (1) and (2) we have $(A \cap B) \cap C = A \cap (B \cap C)$ Proved.

Proof of $A \cap (A \cup B) = A$

Proof : Let $x \in A \cap (A \cup B) \Rightarrow x \in A$ and $x \in (A \cup B) \Rightarrow x \in A \quad \therefore A \cap (A \cup B) \subseteq A$

Conversely let $y \in A$, hence $y \in A \cup B$ and hence $y \in A \cap (A \cup B)$

$\therefore A \subseteq A \cap (A \cup B)$

combining the reverse inclusion we have $A \cap (A \cup B) = A$. Proved.

Proof of $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Proof : Let $x \in S$. Then $x \in \overline{A \cup B} \Rightarrow x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in \overline{A}$ and $x \in \overline{B} \Rightarrow x \in \overline{A} \cap \overline{B}$ Thus $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$

Again let $y \in$ S. Then $y \in \overline{A} \cap \overline{B} \Rightarrow y \in \overline{A}$ and $y \in \overline{B} \Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B) \Rightarrow y \in (\overline{A \cup B})$ Thus $\overline{A} \cap \overline{B} \subseteq (\overline{A \cup B})$

Thus $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq (\overline{A \cup B})$

Hence it follows from definition that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. Proved.

We can also establish the laws or properties by both subset relations simultaneously by using logical equivalence '$\Leftrightarrow$' instead of logical implications ($\Rightarrow$ and $\Leftarrow$).

For example let us prove the second equality of (vi) of distributive law.

Proof of $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof : for each $x \in S$, $\quad x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $(x \in B \cup C)$

$\Leftrightarrow (x \in A)$ and $(x \in B$ or $x \in C) \Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Leftrightarrow x \in A \cap B$ or $x \in A \cap C \Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

As we have equivalent statements throughout, hence we have

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

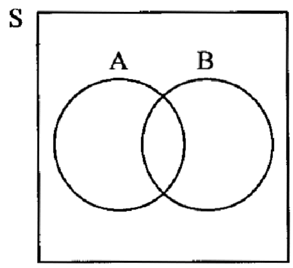
Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Proved.

As in the laws of logic we see in the pair of laws in set theory from (iii) to (x) that first law can be deduced from second by interchanging $\cup$ and $\cap$ and also by interchanging S and ϕ and vice versa. This principle is known as *duality principle* in set theory.

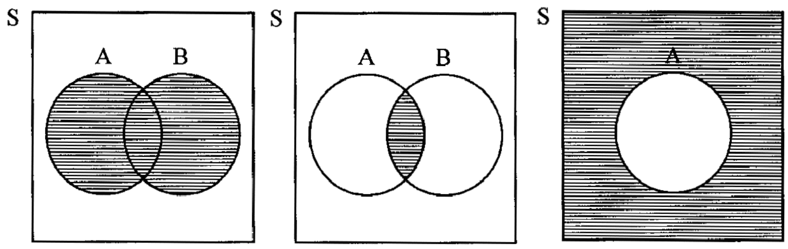
3.3.4 Venn diagrams :

The set operations — union, intersection, complementation etc can be visualised from the diagramatic representations of sets known as Venn diagram. It is a pictorial representation of sets in which sets are represented by enclosed areas in the plane.

The universal set U or S is represented by the interior of a rectangle and the other sets are represented by disks lying within the rectangle. If A and B are two arbitrary sets then it may happen that some objects are in A but not in B, some objects are in B but not in A, some objects are in both A and B and some are neither in A nor in B. Hence in general we represent A and B as in the following figure :

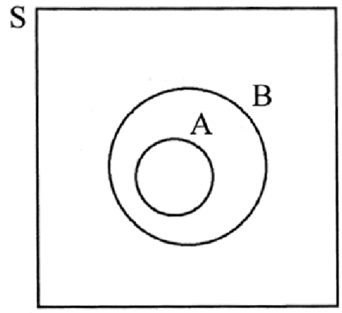


Hence $A \cup B$, $A \cap B$, $\bar{A}$ can be represented by the shaded portion as follows :

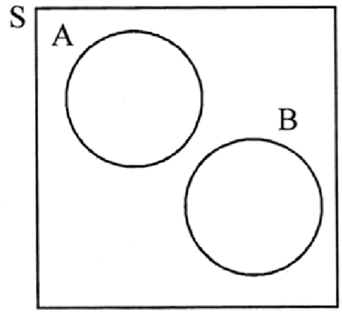


$A \cup B$ $A \cap B$ $\bar{A}$

Moreover if $A \subseteq B$ then the disk representing A will be entirely within the disk representing B. Again if A and B are disjoint then they have no elements in common. These can be represented as follows :



$A \subseteq B$

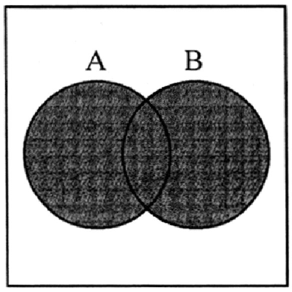


A and B are disjoint

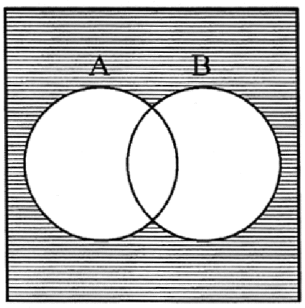
We can also establish the laws of set theory by drawing the Venn diagram.

For example if we want to verify the De Morgan's law $\left(\overline{A \cup B}\right) = \overline{A} \cap \overline{B}$ we proceed as follows :

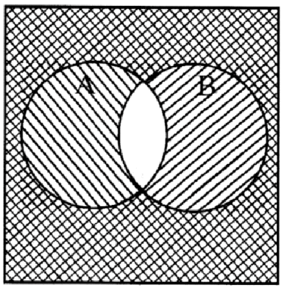
Step 1 : First we draw the Venn diagram for $A \cup B$ which is the shaded area as the following.



Step 2 : Next, we draw th Venn diagram of $\left(\overline{A \cup B}\right)$ which is the shaded area as follows:



Step 3 : Next, we draw the Venn diagram of $\bar{A}$ and $\bar{B}$ by strokes // and \\ respectively and the Venn diagram for $\bar{A} \cap \bar{B}$ which is the crosed shaded area as follows :



Step 4 : Since the shaded area represented in step 2 and crossed shadede area in step 3 are equal hence $\left(\overline{A \cup B}\right) = \bar{A} \cap \bar{B}$

3.4 Finte sets, Infinite sets and counting principles

3.4.1 Finite sets and Infinite Sets :

A set X is called a finite set if it contains exactly n distinct elements where n is a nonnegative integer otherwise it is said to be an infinite set. Thus a finite set is either an empty set or the process of counting of elements surely comes to an end. The number of distinct elements counted in a finite set A is denoted by $n(A)$, called the cardinal number of the set A or cardinality of A. Thus if A be the set of the English alphabets then $n(A) = 26$.

The cardinality of a set A may also be denoted by $|A|$ or $card(A)$. thus the cardinality of empty set ϕ is 0. Again the set of odd opsitive integers is not a finite set because the process of counting of elements of the set of odd positive integers does not come to an end.

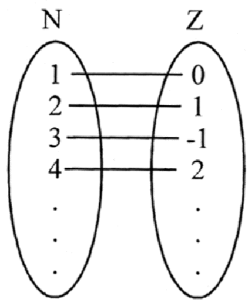
3.4.2 Countable set :

Infinite sets are of two categories, countably infinite and uncountable. A set is said to be countably infinite if there exists an one to one corespondence between the elements in the set A and the elements of N. Thus an infinite set is countably infinite if there exists a bijective mapping from N to A.

For example the set of integers Z is countably infinite since if we define $f : N \to Z$ as

$$f(n)=\begin{cases}\dfrac{n}{2} & \text{if } n \text{ even}\\ -\dfrac{(n-1)}{2} & \text{if } n \text{ odd}\end{cases}$$

then f is obiviously a bijective mapping (see the figure below)



A countable infinite set is also termed as denumerable. A set that is either finite or denumerable is called countable. A set which is not countable is called uncountable or nondenumerable set.

Suppose that A and B are two sets (finite or infinite). We say that A and B have the same cardinality (written as $|A| = |B|$) if a bijective mapping exists between A and B.

Thus by our earlier discession, N and Z have same cardinality.

3.4.3 Counting Principle, The inclusion-exclusion principle :

Sets are extensively used in counting problems— for which we need to study about the size of the sets. For a finite set A, we know that the size of the set A is $n(A)$ or cardinal no. of the set A. Now if A and B are two disjoint finite sets, let us find $n(A \cup B)$. In counting the elements of $A \cup B$, first we count those that are in A which is $n(A)$. Next we count only other elements of $A \cup B$ i.e. the no. of element which are in B but not in A. But since A and B are disjoint hence there are $n(B)$ elements that are B and not in A. Therefore $n(A \cup B) = n(A) + n(B)$.

We can also find a formula for $n(A \cup B)$ even when they are not disjoint. This formula states as follows :

If A and B are two finite sets then $n(A \cup B) = n(A) + n(B) - n(A \cup B)$

Proof : In counting the elements of $A \cup B$ we add $n(A)$ and $n(B)$ then subtract $n(A \cup B)$.

This means we include $n(A)$ and $n(B)$ and then we exclude $n(A \cap B)$.

Because when we add $n(A)$ and $n(B)$ we have counted the elements of $A \cap B$ twice. So to obtain the accurate value of $n(A \cup B)$ we have to subtract $n(A \cap B)$ from $n(A) + n(B)$.

Hence $n(A \cap B) = n(A) + n(B) - n(A \cap B)$

The above principle is the simplest from of Inclusive-Exclusive principle. This principle holds for any number of sets.

For example for three sets it will be as follows :

If A, B, C are three finite sets then

$$n(A \cup B \cup C) = n(A) + n(B) + n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

In this case we first include $n(A)$, $n(B)$, $n(C)$ then we exclude $n(A \cap B), n(B \cap C), n(C \cap A)$ and then we again include $n(A \cap B \cap C)$.

3.5 Classes of sets, Power set of a set, Cartesian product of sets.

3.5.1 Classes of sets :

We have defined a set as a well defined collection of distinct objects or elements. Sets can also contain other sets. For example the set $\{N, Q\}$ is a set containing two infinite sets N and Q. $\{\{1, 2\}, \{5\}\}$ is a set containing two finite sets $\{1, 2\}$ and $\{5\}$. Thus in the set $\{N, Q\}$, N is an element of $\{N, Q\}$ i.e. $N \in \{N, Q\}$. Similarly $\{1, 2\} \in \{\{1, 2\}, \{5\}\}$.

This type of set where the elements are also sets is called a class of sets or a family of sets or set of sets. Sets containing sets arise naturally when an aplication need to consider some or all of the subsets of a base set A(say).

For example let us take $A = \{1,2,3,5,7\}$

Now let B be the class of sets whose elements are subsets of A containing exactly four elements. Then $B = \{\{1,2,3,5\}, \{1,2,3,7\}, \{1,2,5,7\}, \{1,3,5,7\}, \{2,3,5,7\}\}$ Again since A is a set of five elements hence its cardinality $n(A) = 5$. Again B is a set of five elements hence its cardinality 5.

The empty set like any other set can be an element of a set of sets. Thus $\{\phi, \{1,2,\}\}$ is a set containg two elements, the empty set ϕ and the set $\{1,2\}$.

3.5.2 Power set of a set :

The collection of all subsets of a set S is a class of sets. This set is said to be the power set of S and is denoted by $P(S)$.

Thus $P(S) = \{A : A \text{ is a subset of } S\}$

If S is a finite set of n elements then $P(S)$ contains 2^n subsets. This due the fact that any subset of S is either ϕ or a subset containing i elements of S for $i = 1, 2, 3, \ldots.. n$. which implies that the number of elements of $P(S) = 1 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots + {}^nC_n = 2^n$

For example if $A = \{1, 2, 3\}$ then

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

We note that power set of A always contains the empty set ϕ, regardless what is in A. As a consequence $P(\phi) = \{\phi\}$.

3.5.3 Cartesian product of sets :

Let A and B be nonempty sets. The cartesian product of A and B denoted by $A \times B$ is the set defined by $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$.

We assume $A \times \phi = \phi$ and $\phi \times A = A$ for any set A. We observe that if A and B are finite sets such that $n(A)$ or $|A| = m_1$ and $n(B)$ or $|B| = m_2$ then $n(A \times B)$ or $|A \times B| = m_1 m_2$

We can also consider the cartesian product $A \times A$ of a set A itself denoted by A^2.

For example if $A = \{1,2,\}$ $B = \{2, 3\}$ then

$A \times B = \{(1,2), (1,3), (2,2), (2,3)\}$ and $B \times A = \{(2,1), (2,2)\ (3,1), (3,2)\}$

$A^2 = A \times A = \{(1,1), (1,2), (2,1), (2,2)\}$.

3.6 Examples

3.6.1 Let $A = \{2,5,\{5\}\}$ which of the following statements are true ?

(i) $2 \in A$ (ii) $\{2\} \in A$ (iii) $\{5\} \in A$ (iv) $\{5\} \subseteq A$ (v) $\{\{5\}\} \subseteq A$ (vi) $\{2\} \subset A$ (vii) $A = \{2,5\}$

Solution :

(i) $2 \in A$ is true since 2 is an element of A.

(ii) Since {2} is a set containg one element 2 but this set is not an element of A. Hence the statement is false.

(iii) Since A contains the set {5} as an element hence $\{5\} \in A$ is true.

(iv) Since 5 is also an element of A hence the set {5} is a subset of A. Hence the statement is true.

(v) Since the set {5} is also an element of A hence the set {{5}} is a subset of A. Hence the statement is true.

(vi) Since 2 is an element of A hence the set {2} is a subset of of A. But A contains two other elements 5 and {5} hence {2} is a proper subset of A. This means $\{2\} \subset A$ is true.

(vii) Since $\{2,5\} \subseteq \{2,5,\{5\}\}$ but {2,5, {5}} is not a subset of {2,5}. Hence the statement $A = \{2,5\}$ is not true.

3.6.2 Determine all the elements in each of the following.

(i) $A = \{1+(-1)^n / n \in N\}$ (ii) $B = \{2n-1/n \in Z\}$

Solution :

(i) Since $(-1)^n$ is either 1 or – 1 hence $A = \{1+1, 1-1\} = \{2,0\}$

(ii) Since for any $n \in$ Z, $2n - 1$ is an odd integer and any odd integer can be written in the form $2n - 1$ for some $n \in$ Z hence B is the set of all odd integers.

3.6.3 Let $A = \{n \in Z / 0 \le n \le 4\}$ and $B = \{n \in Z / 2 \le n < 7\}$ be two subsets of a universal set $S = \{n \in Z / 0 \le n \le 10\}$

Find (i) $A \cup B$ (ii) $A \cap B$ (iii) $\left(\overline{A \cup B}\right)$ (iv) $\overline{A} \cap \overline{B}$

Solution : Her $A = \{0,1,2,3,4\}$, $B = \{2,3,4,5,6\}$ and $S = \{0,1,2,3,4,5,6,7,8,9,10\}$

(i) $A \cup B = \{0,1,2,3,4,5,6\}$ (ii) $A \cap B = \{2,3,4\}$ (iii) $\overline{A \cup B} = \{7,8,9,10\}$

(iv) Since $\overline{A} = \{5,6,7,8,9,10\}$ and $\overline{B} = \{0,1,7,8,9,10\}$ hence $\overline{A} \cap \overline{B} = \{7,8,9,10\}$

3.6.4 Let *A, B, C* be three subsets of a universal set *S*. Prove that

(i) $(A \cap B) \cup (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) = S$

(ii) $\overline{\left(A\cup B\cup\overline{C}\right)\cap\left(A\cup\overline{B}\cup\overline{C}\right)}=C\cap\overline{A}$

Solution:

(i) $\left(A\cap B\right)\cup\left(A\cap\overline{B}\right)\cup\left(\overline{A}\cap B\right)\cup\left(\overline{A}\cap\overline{B}\right)$

$=\left\{(A\cap B)\cup(A\cap\overline{B})\right\}\cup\left\{(\overline{A}\cap B)\cup(\overline{A}\cap\overline{B})\right\}$ (by associative law)

$=\left\{A\cap(B\cup\overline{B})\right\}\cup\left\{\overline{A}\cap(B\cup\overline{B})\right\}$ [by distributive law]

$=(A\cap S)\cup\left(\overline{A}\cap S\right)$ [by inverse law]

$=A\cup\overline{A}$ [by identity law] = S [by inverse law] Proved.

(ii) $\left(A\cup B\cup\overline{C}\right)\cap\left(A\cup\overline{B}\cup\overline{C}\right)$

$=(A\cup D)\cap(A\cup E)$ where $D=B\cup\overline{C}$ and $E=\overline{B}\cup\overline{C}$

$= A \cup (D \cap E)$ [distributive law]

Now $D\cap E=\left(B\cup\overline{C}\right)\cap\left(\overline{B}\cup\overline{C}\right)$

$=\left(B\cap\overline{B}\right)\cup\overline{C}$ [distributive law] $=\phi\cup\overline{C}$ [inverse law] = $\overline{C}$ [identity law]

$\therefore A\cup(D\cap E)=A\cup\overline{C}$

Hence $\overline{\left(A\cup B\cup\overline{C}\right)\cap\left(A\cup\overline{B}\cup\overline{C}\right)}$

$=\overline{\left(A\cup\overline{C}\right)}=\overline{A}\cap\overline{\overline{C}}$ [by De Morgan's law]

$=\overline{A}\cap C$ [law of double complement or Involution law]

$=C\cap\overline{A}$ [commutative law] Proved.

3.6.5 Among integers 1 to 100, how many of them are

(i) divisible neither by 2 nor by 3 nor by 5 ?

(ii) divisible either by 2 or by 3 but not by both ?

(iii) divisible by 2 but not by 3 nor by 5 ?

(iv) divisible either by 2 or by 3 but not ny 5 ?

Draw a suitable Venn diagram for verification.

Solution :

Here total no. of integers = 100

Let S be the set of integers from 1 to 100. $\therefore$ $n(S) = 100$

Let A, B, C be the set of integers from S which are divisible by 2, 3 and 5 respectively.

$$\therefore\ n(A) = \left[\frac{100}{2}\right] = 50,\ n(B) = \left[\frac{100}{3}\right] = 33,\ n(C) = \left[\frac{100}{5}\right] = 20$$

where $[x]$ denotes the greatest integer not greater than x.

Now $n(A \cap B) = \left[\frac{100}{2.3}\right] = \left[\frac{100}{6}\right] = 16$

$$n(B \cap C) = \left[\frac{100}{3.5}\right] = \left[\frac{100}{15}\right] = 6,\quad n(C \cap A) = \left[\frac{100}{2.5}\right] = \left[\frac{100}{10}\right] = 10$$

$$n(A \cap B \cap C) = \left[\frac{100}{2.3.5}\right] = \left[\frac{100}{30}\right] = 3$$

(i) Numbers not divisible by 2 nor by 3 nor by 5

$= n(S) - n(A \cup B \cup C)$

$= 100 - \{n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)\}$

$= 100 - \{50 + 33 + 20 - 16 - 6 - 10 + 3\}$

$= 100 - (106 - 32) = 100 - 74 = 26$

(ii) Numbers divisible either by 2 or by 3 but not by both

$= n(A \cup B) - n(A \cap B) = n(A) + n(B) - n(A \cap B) - n(A \cap B)$

$= 50 + 33 - 16 - 16 = 51$

(iii) Numbers divisible by 2 but not by 3 nor by 5

$= n(A \cup B \cup C) - n(B \cup C)$

$= 74 - (n(B) + n(C) - n(B \cap C))$

$= 74 - (33 + 20 - 6) = 74 - 47 = 27$

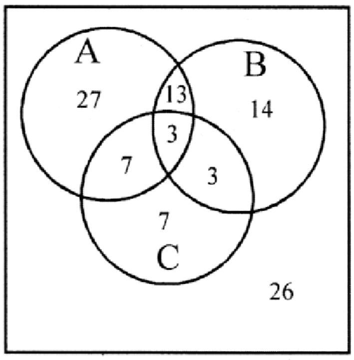
(iv) Numbers divisible either by 2 or by 3 but not by 5

$= n(A \cup B \cup C) - n(C)$

$= 74 - 20 = 54$

All the above results can be verified from the following Venn diagram

$n(S)=100$



3.6.6 If $A=\{x/2\cos^2 x+\sin x\le 2\}$ and $B=\left\{x/\frac{\pi}{2}\le x\le\frac{3\pi}{2}\right\}$

Find $A \cap B$ in simplest form.

Solution :

$2\cos^2 x + \sin x \le 2 \Leftrightarrow 2 - 2\sin^2 x + \sin x \le 2 \Leftrightarrow 2\sin^2 x - \sin x \ge 0$

$\Leftrightarrow \sin x(2\sin x-1)\ge 0 \Leftrightarrow \sin x\left(\sin x-\frac{1}{2}\right)\ge 0$

$\Leftrightarrow$ Either $\sin x \le 0$ or $\sin x \ge \frac{1}{2}$

Thus the set $A = A_1 \cup A_2$ where $A_1=\{x/\sin x\le 0\}$, $A_2=\left\{x/\sin x\ge\frac{1}{2}\right\}$

Hence $A\cap B=(A_1\cup A_2)\cap B$

$=(A_1\cap B)\cup(A_2\cap B)$ [Distributive law]

Now $A_1\cap B=\left\{x/\sin x\le 0 \text{ and } \frac{\pi}{2}\le x\le\frac{3\pi}{2}\right\}=\left\{x/\pi\le x\le\frac{3\pi}{2}\right\}$

Also $A_2\cap B=\left\{x/\sin x\ge\frac{1}{2} \textit{ and } \frac{\pi}{2}\le x\le\frac{3\pi}{2}\right\}=\left\{x/\frac{\pi}{2}\le x\le\frac{5\pi}{6}\right\}$

$\therefore A\cap B=\left\{x/\frac{\pi}{2}\le x\le\frac{5\pi}{6}\right\}\cup\left\{x/\pi\le x\le\frac{3\pi}{2}\right\}$

3.6.7 If $A \cup B = A \cap B$ then show that $A = B$.

Solution :

$A = A \cap (A \cup B)$ (law of absorption)

$= A \cap (A \cap B)$ $(\because A \cup B = A \cap B) = (A \cap A) \cap B$ (Associative law)

$= A \cap B$ (Idempotent law)

Again $B = B \cap (B \cup A)$ (law of absorption)

$= B \cap (A \cup B)$ (commutive law) $= B \cap (A \cap B)$ $(\because A \cup B = A \cap B)$

$= B \cap (B \cap A)$ (commutative law) $= (B \cap B) \cap A$ (Associative law)

$= B \cap A$ (Idempotent law) $= A \cap B$ (Commutative law)

$\therefore$ $A = B$ ***Proved.***

3.6.8 If A, B, C are substes of a universal set S such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$ then prove that $B = C$.

Solution :

$B = (A \cup B) \cap B$ (Absorption law) $= (A \cap C) \cap B$ $(\because A \cup B = A \cup C)$

$= (A \cap B) \cup (C \cap B)$ (Distributive law) $= (A \cap C) \cup (C \cap B)$

$(\because A \cap B = A \cap C)$

$= (A \cap C) \cup (B \cap C)$ (Commutative law) $= (A \cup B) \cap C$ (Distributive law)

$= (A \cup C) \cap C$ $(\because A \cup B = A \cup C)$

$= C$ ***Proved.*** (Absorption law)

3.6.9 Justify the following set-theoretic statements or else give counter examples to disprove.

Let A, B and C be subsets of a set S.

(i) $A \cap B = B \cap C \Rightarrow A = B$

(ii) $A \cup B = B \cup C \Rightarrow A = B$

(iii) $A \cup (B \cap \overline{C}) = (A \cup B) \cap (\overline{A \cup C})$

(iv) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution :

(i) The statement is false.

Let $S = \{1, 2, 3\}$, $A = \{1\}$, $B = \{2\}$, $c = \{3\}$

Then $A \cap C = \phi = B \cap C$ but $A \neq B$

(ii) Let $S = \{1, 2, 3\}$, $A = \{1, 2\}$, $B = \{2\}$, $C = \{1, 2, 3\}$

Then $A \cup C = \{1, 2, 3\} = B \cup C$ but $A \neq B$

Hence the statement is false.

(iii) The statement is false.

Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7\}$ $C = \{7,8\}$

$\therefore \overline{C} = \{1, 2, 3, 4, 5, 6\}$ $\therefore B \cap \overline{C} = \{4, 5, 6\}$

$\therefore A \cup \left(B \cap \overline{C}\right) = \{1, 2, 3, 4, 5, 6\}$

Again $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$, $A \cup C = \{1, 2, 3, 4, 5, 7, 8\}$

$\therefore$ $\overline{A \cup C} = \{6\}$ $\therefore (A \cup B) \cap \overline{A \cup C} = \{6\}$

$\therefore$ $A \cup \left(B \cap \overline{C}\right) \neq \left(A \cup B\right) \cap \overline{A \cup C}$

(iv) The statement is true.

Proof : For any $(x, y) \in A \times (B \cap C)$

$\Leftrightarrow x \in A$ and $y \in (B \cap C) \Leftrightarrow x \in A$ and $y \in B$ and $y \in C$

$\Leftrightarrow (x, y) \in A \times B$ and $(x, y) \in A \times C \Leftrightarrow (x, y) \in (A \times B) \cap (A \times C)$

Now $(x, y) \in A \times (B \cap C) \Rightarrow (x, y) \in (A \times B) \cap (A \times C)$

shows that $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$...............(1)

Again $(x, y) \in (A \times B) \cap (A \times C) \Rightarrow (x, y) \in A \times (B \cap C)$

shows that $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$...............(2)

From (1) and (2) we conclude that

$A \times (B \cap C) = (A \times B) \cap (A \times C)$

3.6.10 Prove that the following three statements are equivalent

(i) $A \subset B$ (ii) $A \cap \overline{B} = \phi$ (iii) $A \cap B = A$

Solution :

Let the statement (i) be true i.e. $A \subset B$

Hence $A \cap \overline{B} = \{x : x \in A \text{ and } x \in \overline{B}\}$

$= \{x : x \in B \text{ and } x \in \overline{B}\}$ $(\because A \subset B)$

$= \{x : x \in B \text{ and } x \notin B\} = \phi$

Hence condition (i) $\Rightarrow$ condition (ii)

Again let $A \cap \overline{B} = \phi$ $\therefore$ For any $x \in A \Rightarrow x \notin \overline{B}$ *i.e.* $x \in A \Rightarrow x \in B$

$\therefore$ $A \subset B$

Hence condition (ii) $\Rightarrow$ condition (i)

Again let $A \subset B$ be true. This means $x \in A \Rightarrow x \in B$ ....(1)

Hence $A \cap B = \{x : x \in A \text{ and } x \in B\} = \{x : x \in A\} = A$ [by(1)]

Hence condition (i) $\Rightarrow$ condition (iii)

Again let $A \cap B = A$ be true.

Then $x \in A \Rightarrow x \in A \cap B \Rightarrow x \in A$ and $x \in B \Rightarrow x \in B$ $\therefore$ $A \subset B$

Hence condition (iii) $\Rightarrow$ condition (i). Hence the proof.

3.7 Exercise

3.7.1 Chose the correct option :

(i) Two finite sets have m and n elements. the number of subsets of the first set is 112 more than that of the second set. The values of m and n are respectively

(a) 4,7, (b) 7,4 (c) 6,4 (d) 8,7

(ii) If $A = \{8^n - 7n - 1 : n \in N\}$ and $B = \{49n - 49 : n \in N\}$ then

(a) $A \subseteq B$ (b) $B \subseteq A$ (c) $A = B$ (d) none of those

(iii) The set $\left(\overline{A \cap \overline{B}}\right) \cup (B \cap C)$ is equal to

(a) $\overline{A} \cup B \cup C$ (b) $\overline{A} \cup B$ (c) $\overline{A} \cup \overline{C}$ (d) $\overline{A} \cap B$

(iv) Let S be the universal set for sets A and B. If $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$ then $n(\overline{A} \cap \overline{B}) = 300$ provided $n(S)$ is equal to

(a) 600 (b) 700 (c) 800 (d) 900

(v) In a school of 300 students, every student reads 5 newspaper and every newspaper is read by 60 students. Then the number of newspaper is

(a) at least 30 (b) atmost 20 (c) exactly 25 (d) none of those

(vi) Let A and B be two nonempty subsets of a universal set S such that A is not a subset of B, then

(a) $A \subseteq \overline{B}$ (b) $B \subseteq A$ (c) $A \cap B = \phi$ (d) $A \cap \overline{B} \neq \phi$

3.7.2 Use the laws of sets to prove that

(i) $A \cup B = (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (A \cap B)$

(ii) $(A \cap B \cap C) \cup (A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C) \cup (A \cap \overline{B} \cap \overline{C}) = A$

3.7.3 Let S be a universal set and A be a fixed subset of S.

(i) If $A \cup B = B$ holds for all subset B of S, prove that $A = \phi$.

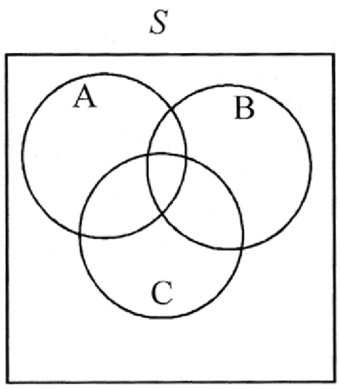
(ii) If $A \cap B = B$ holds for all subsets B of S, prove that $A = S$.

3.7.4 Find the number of subsets of the set S = {1,2,3,4,..........15} such that the product of the elements in each subset is a multiple of 3.

3.7.5 Let A, B and C be three subsets of a universal set S such that $A \cup B = A \cup C$ and $\overline{A} \cup B = \overline{A} \cup C$, prove that $B = C$.

3.7.6 Find the power set of the set A where A = {{1,}, {2}, {1,2}}.

3.7.7 The following Ven diagram shows the subsets A, B, C of a universal set S.



Shade the following subsets of S

(i) $A \cap \overline{(B \cup C)}$ (ii) $\overline{A} \cap (B \cup C)$ (iii) $\overline{A} \cup (C \cap \overline{B})$

3.7.8 Let A and B be two subsets of a universal set S. Prove that $P(A \cap B) = P(A) \cap P(B)$ where $P(A)$ denotes the power set of A.

Give an example of two subsets A and B of a universal set S such that $P(A \cup B) \neq P(A) \cup P(B)$.

3.7.9 Write the dual statement of the following set-theoretic result $A = (A \cup B) \cap (A \cup \phi)$.

3.7.10 Is it possible to draw a Venn diagram of sets A, B, C where $A \subseteq B$, $B \cap C = \phi$ and $A \cap C \neq \phi$? Justify.

3.8 Answers to the exercise 3.7

3.7.1 (i) (b) $\left[\text{Hints}: 2^m - 2^n = 112 = 2^4(2^3 - 1)\right]$

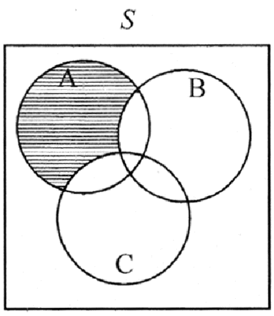
(ii) (a) (iii) (b) (iv) (b) (v) (c) (vi) (d)

3.7.4 $2^{10}(2^5 - 1)$

3.7.6 $P(A) = \{\phi, \{\{1\}\}, \{\{2\}\}, \{\{1, 2\}\}, \{\{1\}, \{2\}\}, \{\{1\}, \{1, 2\}\}, \{\{2\}, \{1, 2\}\}, \{\{1\}, \{2\}, \{1, 2\}\}\}$

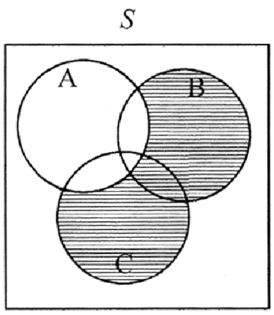
3.7.7

(i)



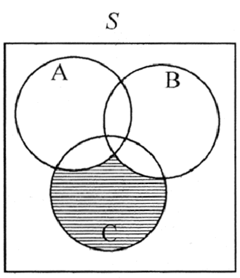
$A \cup \overline{(B \cup C)}$

(ii)



$\bar{A} \cup (B \cup C)$

(iii)



$\bar{A} \cap (C \cap \bar{B})$

3.7.8 2nd Part :

Let $S = \{1,2,3\}$, $A = \{1\}$, $B = \{2\}$

Then $\{1,2\} \in P(A \cup B)$ but $\{1,2\} \notin P(A) \cup P(B)$

3.7.9 $A = (A \cap B) \cup (A \cap S)$ where S is the universal set.

3.7.10 No such Venn diagram exists.

Because $A \cap C \neq \phi \Rightarrow \exists x\, s.t.\ x \in A$ and $x \in C \Rightarrow x \in B$ and $x \in C$ $(\because A \subseteq B)$

$\Rightarrow x \in B \cap C \Rightarrow B \cap C \neq \phi$

3.9 Summary

In this unit we have learnt the basic concept of set theory which is one of the most fundamental in mathematics. Several mathematical concepts specially on mathematical analysis, abstract algebra, discrete mathematics and topology can be defined precisely using only set theoretic concepts.

Unit - 4 ❑ Difference and Symmetric difference of two sets. Set Identities

Structure

4.0 Objectives

After going through this unit the learner should be able to:

- Perform two more operations, difference and symmetric difference of two sets.
- Understand the set identities and the various method of proof.
- Know the generalised De Morgan's laws.

4.1 Introduction

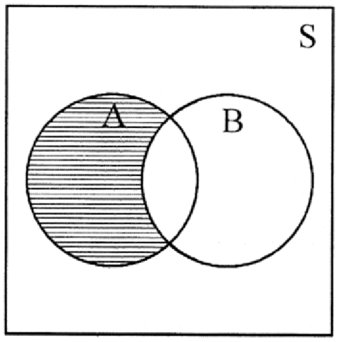
Set theory uses a number of different operations to construct a new set from old ones. It is important to have well defined ways to construct these new sets and example of these include union, intersection of two sets and complement of a set. In this unit we shall learn about two more operations difference and symmetic difference of two sets.

4.2 Difference and symmetric difference

4.2.1 Difference and symmetric difference of two sets :

The difference of two subsets A and B of a universal set S is denoted by $A\backslash$B or A-B is the set of elements which belong to A but which do not belong to B. This means A-B or $(A \backslash B) = \{x \in S : x \in A \text{ and } x \notin B\}$ or simply = $\{x : x \in A$ and $x \notin B\}$

The shaded portion of the following Venn diagram represents the set $A\backslash B$.



A-B is shaded

The subset A-B of S is also called complement of B relative to A.

The difference A-B can also be expressed as $A - B = A \cap \overline{B}$ where $\overline{B}$ is the complement of B.

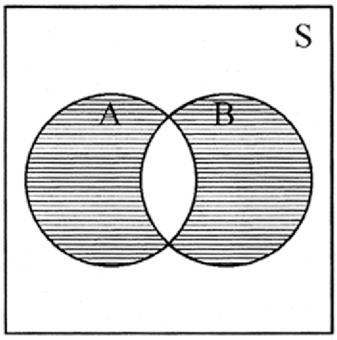
In particular $A - B = \phi \Leftrightarrow A \subseteq B$ and $A - B = A \Leftrightarrow A \cap B = \phi$

For example if $S = \{1,2,3,4,5\}$, $A = \{1,2,3\}$, $B = \{3,4,5\}$ then $A - B = \{1,2\}$, $B - A = \{4,5\}$

The symmetric difference of two subsets A and B of a universal set S is denoted by $A \Delta B$ or $A \oplus B$ consists of those element which belong to A or B but not to both.

That is $A \Delta B = (A \cup B) - (A \cap B)$.........(1). $A \Delta B$ can also be equivalently expressed as $A \Delta B = (A - B) \cup (B - A) = (A \cap \overline{B}) \cup (B \cap \overline{A})$ .....(2). The equivalence of two definitions (1) and (2) will be proved in the next article.

The shaded portion of the following Venn diagram represents the set $A \Delta B$



A Δ B is shaded

The name symmetric difference suggests a connection with the difference of two sets which is evident from both the formulas (1) and (2). The symmetric difference is also known as disjunctive union of two sets since it is the union of two disjoint subsets A-B and A-B provided A-B and B-A are both nonempty. Again by construction the role of A and B can be changed. Thus $A \Delta B = B \Delta A$.

For example if $S = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3\}$ $B = \{3, 4, 5\}$

then $A \Delta B = (A - B) \cup (B - A) = \{1, 2\} \cup \{4, 5\} = \{1, 2, 4, 5\}$

Again it follows from definition that for any set A,

$A \Delta A = (A \cup A) - (A \cap A) = A - A = \phi$

and $A \Delta \phi = (A \cup \phi) - (A \cap \phi) = A - \phi = A$

4.2.2 Properties of difference and symmetric difference:

Some important properties of difference and symmetric difference are as follows:

For any sets *A*, *B*, *C* taken from a universal set *S* we have

(i) $A - (B \cap C) = (A - B) \cup (A - C)$

$A - (B \cup C) = (A - B) \cap (A - C)$

(ii) $A \Delta B = B \Delta A$ (Commutativity)

(iii) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$ (Associativity)

<u>Proof of</u> $A-(B\cap C)=(A-B)\cup(A-C)$ [that is first part of 4.1.2 (i)]

Proof : $A-(B\cap C)=A\cap\left(\overline{B\cap C}\right)=A\cap\left(\overline{B}\cup\overline{C}\right)$

$=(A\cap\overline{B})\cup(A\cap\overline{C})$ (by distributive law)

$=(A-B)\cup(A-C)$ Proved

<u>Proof of</u> $A-(B\cup C)=(A-B)\cap(A-C)$ [that is second part of 4.1.2(i)]

Proof : $A-(B\cup C)=A\cap\overline{(B\cup C)}$ (by definition)

$=A\cap(\overline{B}\cap\overline{C})$ by De Morgan's law

$=A\cap\overline{B}\cap\overline{C}=(A\cap\overline{B})\cap(A\cap\overline{C})=(A-B)\cap(A-C)$ **Proved**.

<u>Proof of (ii)</u> $A\Delta B=B\Delta A$

Proof : By definition $A\Delta B=(A\cup B)-(A\cap B)$

$=(B\cup A)-(B\cap A)$ (by commutative law for union & intersection)

$=B\Delta A$ (by definition) **Proved.**

<u>Proof of (iii)</u> $A\Delta(B\Delta C)=(A\Delta B)\Delta C$

Proof : By definition $B\Delta C=\left(B\cap\overline{C}\right)\cup(\overline{B}\cap C)$

$\therefore A\Delta(B\Delta C)$

$=\left(A\cap\overline{(B\Delta C)}\right)\cup\left(\overline{A}\cap(B\Delta C)\right)$

$=\left[A\cap\left\{\overline{\left(B\cap\overline{C}\right)}\cap\overline{\left(\overline{B}\cap C\right)}\right\}\right]\cup\left[\overline{A}\cap\left\{\left(B\cap\overline{C}\right)\cup\left(\overline{B}\cap C\right)\right\}\right]$

$=\left[A\cap\left\{\left(\overline{B}\cup C\right)\cap\left(B\cup\overline{C}\right)\right\}\right]\cup\left[\left\{\overline{A}\cap\left(B\cap\overline{C}\right)\right\}\cup\left\{\overline{A}\cap\left(\overline{B}\cap C\right)\right\}\right]$

$=\left[A\cap\left\{\left(\overline{B}\cap B\right)\cup\left(\overline{B}\cap\overline{C}\right)\cup\left(C\cap B\right)\cup\left(C\cap\overline{C}\right)\right\}\right]\cup\left(\overline{A}\cap B\cap\overline{C}\right)\cup\left(\overline{A}\cap\overline{B}\cap C\right)$

$=\left[A\cap\left\{\left(\overline{B}\cap\overline{C}\right)\cup\left(B\cap C\right)\right\}\right]\cup\left(\overline{A}\cap B\cap\overline{C}\right)\cup\left(\overline{A}\cap\overline{B}\cap C\right)$

$= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap B \cap C) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$

$\therefore A \Delta (B \Delta C)$

$= (A \cap \overline{B} \cap \overline{C}) \cup (A \cap B \cap C) \cup (\overline{A} \cap B \cap \overline{C}) \cup (\overline{A} \cap \overline{B} \cap C)$..........(1)

Interchanging A and C in (1) we have

$C \Delta (B \Delta A)$

$= (C \cap \overline{B} \cap \overline{A}) \cup (C \cap B \cap A) \cup (\overline{C} \cap B \cap \overline{A}) \cup (\overline{C} \cap \overline{B} \cap A)$

$= (\overline{A} \cap \overline{B} \cup C) \cup (A \cap B \cap C) \cup (\overline{A} \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap \overline{C})$..........(2)

From (1) and (2) we have

$A\Delta(B\Delta C) = C\Delta(B\Delta A) = (B\Delta A)\Delta C \quad (\because A\Delta B = B\Delta A)$

$= (A\Delta B)\Delta C$ **Proved.**

Let us now prove the equivalence of two definitions (equations (1) and (2) in 4.1.1) of symmetric difference of two sets A and B. (*i.e.*, $A \Delta B$) as follows :

<u>**Proof of**</u> $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

Proof : $(A \cup B) - (A \cap B) = (A \cup B) \cap \overline{(A \cap B)}$

$= (A \cup B) \cap (\overline{A} \cup \overline{B})$ (by De Morgan's law)

$= \{(A \cup B) \cap \overline{A}\} \cup \{(A \cup B) \cap \overline{B}\}$ (by distributive law)

$= (A \cap \overline{A}) \cup (B \cap \overline{A}) \cup (A \cap \overline{B}) \cup (B \cap \overline{B})$ (by distributive law)

$= \phi \cup (B - A) \cup (A - B) \cup \phi \quad (\because X \cap \overline{X} = \phi)$

$= (B - A) \cup (A - B) \quad (\because X \cup \phi = X) = (A - B) \cup (B - A)$ (by commutative law)

Hence $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$ Proved.

4.3 Set indentities and the various methods of proof

Set identities are methods of expressing the same set using the names of set and set operations. They can be used in the algebra of sets. We have already discussed about the laws of set theory in 3.2.3 for any sets A, B, C taken from universal set S, those laws are set identities. Also we have proved properties of difference and symmetric difference in 4.1.1. Those are also set identities. Any identity relating to sets can be proved in many ways. As for example let us prove De Morgan's law $\overline{A \cup B} = \overline{A} \cap \overline{B}$ for any sets A, B taken from a universal set S in various methods.

1st method In this method we shall show that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and $\overline{A} \cap \overline{B} \subseteq \overline{A \cup B}$.

This method is known as double inclusion method. This method of proof is already discussed in 3.2.3

2nd method In this method we use the logical equivalence of defining propositions or by using quantified definition of union, intersection or complement

Proof : $\overline{(A \cup B)} = \{x \in S : \sim (x \in A \vee x \in B)\}$ where S is the universal set.

$= \{x \in S : (\sim x \in A) \wedge (\sim x \in B)\} = \{x \in S : (x \in \overline{A}) \wedge (x \in \overline{B})\}$

$= \{x \in S : (x \in \overline{A} \cap \overline{B})\} = \overline{A} \cap \overline{B}$ Proved.

3rd Method In this method we use a membership table. This method works the same way as a truth table in which we use T and F to indicate the following :

T : For x is an element belongs to the specified set

F : For x is an element does not belong to the specified set

Proof : Let us prepare the membership table for $\overline{A \cup B}$ and $\overline{A} \cap \overline{B}$

A	B	$\overline{A}$	$\overline{B}$	$A \cup B$	$\overline{A \cup B}$	$\overline{A} \cap \overline{B}$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the last two column we have $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$

4.4 Generalised union and intersections

We now generalise the concepts of union and intersection of two sets to an arbitrary collection of sets. Let us consider a finite collection of n sets A_1, A_2, ...A_n ($n \geq 2$)

In this case we write

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 .... \cup A_n = \{x : x \in A_i \text{ for some } i, 1 \leq i \leq n\}$$

and $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap A_3 .... \cap A_n = \{x : x \in A_i \textit{ for all } i, 1 \leq i \leq n\}$

To generalise it further for any arbitrary family of sets, finite or infinite we introduce the notion of an index set. A nonempty set I is said to be an index set for a family F of sets, if for any $i \in I$ there exists a set $A_i \in F$ and $F = \{A_i : i \in I\}$. It can be noted that I can be a finite or infinite set. We now define the union and intersection of the sets A_i, $i \in I$, as follows :

$$\bigcup_{i \in I} A_i = \{x : x \in A_i \text{ for at least one } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x : x \in A_i \text{ for all } i \in I\}$$

The above definition of the union and intersection of the family of subsets can be restated by using quantifiers as follows :

$x \in \bigcup_{i \in I} A_i \Leftrightarrow (\exists i \in I)(x \in A_i)$ and $x \in \bigcap_{i \in I} A_i \Leftrightarrow (\forall i \in I)(x \in A_i)$

Thus $x \notin \bigcup_{i \in I} A_i \Leftrightarrow \sim [(\exists i \in I)(x \in A_i)]$

$$\Leftrightarrow (\forall i \in I)(x \notin A_i) \Leftrightarrow (\forall i \in I)(x \in \overline{A_i})$$

Thus $\overline{\left(\bigcup_{i \in I} A_i\right)} = \bigcap_{i \in I} \overline{A_i}$

This is generalised De Mogran's law. In a similar manner we see that

$$x \notin \bigcap_{i \in I} A_i$$

$$\Leftrightarrow \sim \left[(\forall i \in I)(x \in A_i)\right]$$

$$\Leftrightarrow (\exists i \in I)(x \notin A_i)$$

$\Leftrightarrow (\exists i \in I)(x \in \overline{A_i})$

Hence $\overline{\left(\bigcap_{i \in I} A_i\right)} = \bigcup_{i \in I} \overline{A_i}$

This is another generalised De Morgan's law.

4.5 Examples :

4.5.1. Let $A = \{1,2,3,4\}$, $B = \{3,4,5,6\}$, $D = \{1,5,7\}$, $E = \{5,7\}$, $F = \{2,4,6\}$ be subsets of a universal set $S = \{1,2,3,4,5,6,7,8\}$. Find

i) A–B ii) B-A iii) $A \Delta B$ iv) D-E v) E-D vi) $D \Delta E$ vii) $D \Delta F$ viii) $\overline{(A - F)}$

ix) $\overline{(B \Delta D)}$ x) $A \Delta (B \Delta F)$

Solution :

i) A-B = The set consists of elements in A which do not belong to B = {1,2,}

ii) B-A = {5,6} iii) $A \Delta B = (A - B) \cup (B - A) = \{1, 2, 5, 6\}$

iv) D-E = {1} v) $E - D = \phi$

vi) $D \Delta E = (D - E) \cup (E - D) = \{1\} \cup \phi = \{1\}$

vii) $D \Delta F$ = the set consists of elements in D or in F but not in both D and F = {1,2,3,4,5,6,7}

viii) A-F = {1,3} $\therefore \overline{(A - F)} = \{2, 4, 5, 6, 7, 8\}$

ix) B D D = {3,4,6,1,7} $\therefore \overline{(B \Delta D)} = \{2, 5, 8\}$

x) A = {1,2,3,4} B D F = {3,5,2} $\therefore$ $A \Delta (B \Delta F)$ = {1,4,5}

4.5.2. For any three subsets A, B, C of a universal set S prove that

i) $(A - C) \cap (B - C) = (A \cap B) - C$

ii) $[(A - B) \cup B = A] \Leftrightarrow B \subseteq A$

iii) $A - B = A - (A \cap B) = (A \cup B) - B$

iv) $(A \Delta B = A \Delta C) \Rightarrow B = C$

v) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

Solutions :

i) $(A-C)\cap(B-C)=(A\cap\overline{C})\cap(B\cap\overline{C})$

$=(A\cap\overline{C})\cap(\overline{C}\cap B)$ (by commutative law) = $\{(A\cap\overline{C})\cap\overline{C}\}\cap B$

(by associative law)

$=\{A\cap(\overline{C}\cap\overline{C})\}\cap B$ (by associative law)

$=(A\cap\overline{C})\cap B$ $(\because \overline{C}\cap\overline{C}=\overline{C})$

$=A\cap(B\cap\overline{C})$ (by commutative law) = $(A\cap B)\cap\overline{C}$ (by associative law)

$=A\cap B-C$ ***Proved.***

ii) $(A-B)\cup B=(A\cap\overline{B})\cup B$

$=(A\cup B)\cap(\overline{B}\cup B)$ (by distributive law) = $(A\cup B)\cap S$ (by inverse law)

= $A\cup B$ (by identity law)

$\therefore$ $(A-B)\cup B=A\cup B$

Thus $[(A-B)\cup B=A]\Leftrightarrow[(A\cup B)=A]$

Hence we have to prove $[(A\cup B)=A]\Leftrightarrow B\subseteq A$

Now if $[(A\cup B)=A]$ then $B\subseteq A\cup B=A\Rightarrow B\subseteq A$. Proved.

Conversely if $B\subseteq A$ then $A\cup B=A$ Proved.

iii) $A-(A\cap B)=A\cap\overline{(A\cap B)}$

$=A\cap(\overline{A}\cup\overline{B})$ (by De Morgan's law)

= $(A\cap\overline{A})\cup(A\cup\overline{B})$ (by distributive law)

$=\phi\cup(A\cap\overline{B})$ $(\because A\cap\overline{A}=\phi)$

$=A\cap\overline{B}$ (by identity law) = $A-B$ Proved.

Again in a similar manner we have

$(A\cup B)-B=(A\cup B)\cap\overline{B}=(A\cap\overline{B})\cup(B\cap\overline{B})=(A\cap\overline{B})\cup\phi=A-B$ Proved.

$\therefore A-B=A-(A\cap B)=(A\cup B)-B$

iv) We know that $A \Delta A = \phi$ and $A \Delta \phi = A$

Thus $B = B \Delta \phi$

$= B\Delta(A \Delta A)$ $(\because A \Delta A = \phi)$

$= (B \Delta A) \Delta A$ by associative law for symmetric difference

$= (A \Delta B) \Delta A$ by commutative law for symmetric difference

$= (A \Delta C) \Delta A$ $(\because A \Delta B = A \Delta C)$

$= (C \Delta A) \Delta A$ by commutative law for symmetric difference

$= C \Delta (A \Delta A)$

$= C \Delta \phi = C$ Proved.

<u>Alternative proof</u> : Let $x \in B$, Now two cases may arise.

<u>Case 1</u> : $x \in A$

Since $x \in A$ and $x \in B$ hence $x \notin (A \Delta B)$

$\Rightarrow x \notin A \Delta C$ $(\because A \Delta B = A \Delta C)$

$\Rightarrow x \in C$ $(\because x \in A)$

<u>Case 2</u> : $x \notin A$

Since $x \notin A$ and $x \in B$ hence $x \in (A \Delta B)$

$\Rightarrow x \in (A \Delta C) \Rightarrow x \in C$ $(\because x \notin A)$

Thus from all the cases we have $x \in B \Rightarrow x \in C$ $\therefore B \subseteq C$

In a similar argument we can show that $C \subseteq B$. Hence B = C

v) L.H.S. $= A \cap (B \Delta C)$

$= A \cap \{(B - C) \cup (C - B)\}$

$= A \cap \{(B \cap \overline{C}) \cup (\overline{B} \cap C)\}$

$= \{A \cap (B \cap \overline{C})\} \cup \{A \cap (\overline{B} \cap C)\}$ by distributive law

$= (A \cap B \cap \overline{C}) \cup (A \cap \overline{B} \cap C)$ by associative law

Again R.H.S. $= (A \cap B) \Delta (A \cap C)$

$= \{(A \cap B) - (A \cap C)\} \cup \{(A \cap C) - (A \cap B)\}$

$= \{(A \cap B) \cap (\overline{A \cap C})\} \cup \{(\overline{A \cap B}) \cap (A \cap C)\}$

$= \{(A \cap B) \cap (\overline{A} \cap \overline{C})\} \cup \{(\overline{A} \cup \overline{B}) \cap (A \cap C)\}$ by De Morgan's law

$= [\{(A\cap B)\cap \overline{A}\}\cup\{(A\cap B)\cap\overline{C}\}]\cup[\{(\overline{A}\cup\overline{B})\cap A\}\cap C]$ by distributive and associative laws

$=\phi\cup(A\cap B\cap\overline{C})\cup[\{(\overline{A}\cap A)\cup(\overline{B}\cap A)\}\cap C]$ by distributive law

$=(A\cap B\cap\overline{C})\cup[\{\phi\cup(A\cap\overline{B})\}\cap C]$

$=(A\cap B\cap\overline{C})\cup(A\cap\overline{B}\cap C)$

Hence $A\cap(B\Delta C)=(A\cap B)\Delta(A\cap C)$

4.5.3. Let *A, B, C* be subsets of a universal set *S* such that *A*Δ*B* = *C*. Prove that

(i) $A = B\,\Delta\, C$ (ii) $A\cap(B\cup C)=A$

Solution :

(i) $B\Delta C = B\Delta(A\Delta B)$ $(\because A\Delta B = C)$

$= B\Delta(B\Delta A)$ (by commutative property)

$=(B\Delta B)\Delta A$ (by associative property)

$=\phi\Delta A$ $(\because B\Delta B=\phi)$

$=A$ $(\because \phi\Delta A = A)$

(ii) $A\cap(B\cup C)$

$=(A\cap B)\cup(A\cap C)$ (by distributive law)

$=(A\cap B)\cup\{A\cap(A\Delta B)\}$ $(\because A\Delta B = C$ given)

Now $A\cap(A\Delta B)$

$=A\cap\{(A\cap\overline{B})\cup(\overline{A}\cap B)\}$

$=\{A\cap(A\cap\overline{B})\}\cup\{A\cap(\overline{A}\cap B)\}$ by distributive law

$=\{(A\cap A)\cap\overline{B}\}\cup\{(A\cap\overline{A})\cap B\}$ by associative law

$=(A\cap\overline{B})\cup(\phi\cap B)$ $(\because A\cap\overline{A}=\phi)=(A\cap\overline{B})\cup\phi = A\cap\overline{B}$

Hence $A\cap(B\cup C)=(A\cap B)\cup(A\cap\overline{B})$

$=A\cap(B\cup\overline{B})$ by distributive law

$=A\cap S$ $(\because B\cup\overline{B}=S)$

$=A$

Hence $A\cap(B\cup C)=A$ Proved.

4.5.4. Let R be the universal set. Consider the following family F of sets indexed by N as follows :

$F = \{A_i : i \in N\}$ where $A_i = \left\{x \in R : 0 < x < \frac{1}{i} \text{ for all } i \in N\right\}$

Prove that

i) $\bigcup_{i=1}^{\infty} A_i = \{x \in R / 0 < x < 1\}$ ii) $\bigcap_{i=1}^{\infty} A_i = \phi$

iii) $\bigcup_{i=1}^{\infty} \overline{A} = R$ iv) $\bigcap_{i=1}^{\infty} \overline{A} = \{x \in R / x \le 0\} \cup \{x \in R / x \ge 1\}$

Solution :

It is given that

$A_1 = \{x \in R / 0 < x < 1\}$, $A_2 = \left\{x \in R / 0 < x < \frac{1}{2}\right\}$

$A_n = \left\{x \in R / 0 < x < \frac{1}{n}\right\}$ *etc.*

i) First we shall show that for each $n \in N$

$A_n \subseteq A_1$

For $n = 1$, $A_n \subseteq A_1$ is obviously true.

Now for $n > 1$ let $x \in A_n \Rightarrow 0 < x < \frac{1}{n}$

Since $n > 1$ hence $\frac{1}{n} < 1$ $\therefore\ 0 < x < \frac{1}{n} \Rightarrow 0 < x < 1$

Hence $x \in A_1$

Thus $A_n \subseteq A_1$ for all $n \in N$ $\therefore\ \bigcup_{i=1}^{\infty} A_i \subseteq A_1$

Conversely suppose $x \in A_1$ hence by the definition of union $x \in \bigcup_{i=1}^{\infty} A_i$

Hence $A_1 \subseteq \bigcup_{i=1}^{\infty} A_i$

Hence $\bigcup_{i=1}^{\infty} A_i = A_1 = \{x \in R / 0 < x < 1\}$ Proved.

ii) If possible let $\bigcap_{i=1}^{\infty} A_i \neq \phi$ and let $y \in \bigcap_{i=1}^{\infty} A_i$

By the definition of intersection we have $0 < y < \frac{1}{n}$ for every positive integer n.

We know that $\lim_{n \to \infty} \frac{1}{n} = 0$

Thus by the definition of null sequence for any $\in = y$ (say) >0 there exists a positive integer m such that

$\left|\frac{1}{n}-0\right| < \in = y$ for all $n \geq m$

This means $\frac{1}{n} < y$ for all $n \geq m$

This means $0 < \frac{1}{n} < y$ for all $n \geq m$

Thus $y \notin A_n$ for $n \geq m$ $\left(\because A_n = \left\{x \in R / 0 < x < \frac{1}{n}\right\}\right)$

The above result contradicts the fact that $0 < y < \frac{1}{n}$ for every positive integer n.

Hence $\bigcap_{i=1}^{\infty} A_i = \phi$ Proved.

iii) By generalised De Morgan's law we have

$\bigcup_{i=1}^{\infty} \overline{A_i} = \overline{\left(\bigcap_{i=1}^{\infty} A_i\right)} = \overline{\phi} \quad \left(\because \bigcap_{i=1}^{\infty} A_i = \phi\right) = R$ Proved.

iv) By generalised De Morgan's law we have

$\bigcap_{i=1}^{\infty} \overline{A_i} = \overline{\left(\bigcup_{i=1}^{\infty} A_i\right)} = \overline{A_1} \quad \left(\because \bigcup_{i=1}^{\infty} A_i = A_1\right) = R - A_1$

$= \{x \in R / x \notin A_1\} = \{x \in R / x \leq 0 \text{ or } x \geq 1\}$

$= \{x \in R / x \leq 0\} \cup \{x \in R / x \geq 1\}$ Proved.

4.6 Exercise

4.6.1. Let $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}\right\}$, $B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}\right\}$, $C = \left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ **and suppose that the universe is** $S = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}\right\}$

Find (i) $\overline{B-C}$ (ii) $A \Delta B$ (iii) $n(A-B)$ (iv) $\overline{(A \Delta B \Delta C)}$ (v) $n(B) - n(A)$

(vi) $n(B-A)$ (vii) $\overline{A \Delta B}$ (viii) $\overline{A} \Delta \overline{B}$

4.6.2. Let $S = \{x \in R \,/\, 0 \le x \le 1\}$ is the universal set and

$$A_i = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\ldots\ldots, \frac{1}{i}\right\} \text{ for all } i \in N$$

Applying generalised De Morgan's law prove that $\bigcup_{i \in N} \overline{A_i} = \{x \in R / 0 \le x < 1\}$ and

$$\bigcap_{i \in N} \overline{A_i} = \left\{x \in R / 0 \le x \le 1 \; but \; x \ne \frac{1}{n}, n \in N\right\}$$

4.6.3. For any three subsets A, B, C of a universal set S prove that

i) $A \cup B = (A \cap B) \Delta (A \Delta B)$

ii) $(A \,\Delta\, B = C) \Rightarrow C \,\Delta\, A = B$

iii) $(A - B) \cup (B - C) \cup (C - A) = (A \cup B \cup C) - (A \cap B \cap C)$

iv) $A = B \Leftrightarrow A \,\Delta\, B = \phi$

4.6.4. State and prove generalised De Morgan's laws.

4.6.5. Let A, B, C be any three subsets of a universal set S. Prove the following set-theoretic statements if you find them correct or else give counter examples.

(i) $\overline{(A - B)} = \overline{(B - A)}$

(ii) $n(A - B) = n(A) - n(B)$, $n(A)$ being the cardinality of the set A.

(iii) $(A - C = B - C) \Leftrightarrow (A \cup C = B \cup C)$

(iv) $(A - B) - C = A - (B \cup C)$

(v) $A - (A - B) = A \cap B$

(vi) $(A \cup B) - A = A - B$

(vii) $(A - C) - (B - C) = (A - B) - C$

4.7 Answer to the exercise 4.6

4.6.1. i) $\overline{B - C} = \left\{1, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}\right\}$

ii) $A \Delta B = \left\{1, \frac{1}{3}, \frac{1}{6}\right\}$

iii) $n(A - B) = 2$

iv) $\overline{(A \Delta B \Delta C)} = \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{7}\right\}$

v) $n(B) - n(A) = -1$

vi) $n(B - A) = 1$

vii) $\overline{A \Delta B} = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}\right\}$

viii) $\overline{A} \Delta \overline{B} = \left\{1, \frac{1}{3}, \frac{1}{6}\right\}$

4.6.5. All are correct except (i) and (vi), for counter example take S = {1,2,3,4,5}, A = {1,2,3}, B = {2,3,4}

4.8 Summary

In this unit we have discussed about the difference and symmetric difference of two sets. We know that in set theory 'or' represents the union of two sets when used in an inclusive sense. On the otherhand in set theory, if 'or' is used in an exclusive sense then the symmetric difference of two sets can be obtained. Moreover we have seen that the difference between two sets is not symmetric but the symmetric difference between two sets is always symmetric.

Unit - 5 ❑ Relation

Structure

5.0 Objectives

After going through this unit the learner should be able to:

- Understand the term 'binary relation' between two nonempty sets.
- Perform composition of relations.
- Learn various kinds of relations.

5.1 Introduction

In this unit we are going to define relations formally, first binary relation then in general n-ary relation. A relation in everyday life shows an association of objects of a set with objects of other set (or the same set) such as 5 is a divisor of 20, A is a subset of B, 11 is less than 15, Rita has a Maruti car etc. The essence of relation is these associations. In certain sense these relations consider the existence or nonexistence of a certain connection between pair of objects taken in a definite order. We define a relation in terms of these order pairs.

5.2 Product set, Binary relation

5.2.1 Cartesian product, binary relation, terminolgies :

We know that if A and B are two nonempty sets then the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$ is known as the cartesian product of two sets A and B and is denoted by A × B. A binary relation ρ from a set A into a set B is a subset of A × B. Thus any subset ρ of the cartesian product A×B is a relation from A into B. If the order pair $(a,b) \in \rho$ then we say that a is related to b and denoted it by $a\rho b$. On the other hand if $(a,b) \notin \rho$ then we say that a is not related to b and we denote it as $a\bar{\rho}b$ or $a\not\rho b$. Sometimes ρ may be a relation from a set A to itself i.e. $\rho \subseteq A \times A$ and then we speak of relation on A. Thus a binary relation ρ on a set A is a rule that associates some or all elements of A with some or all elements of itself.

For example let $A = \{1,2,3\}$, $B = \{l,m,n\}$ and $\rho = \{(1,m), (2,n)\}$ clearly $\rho \subseteq A \times B$. Hence it is a relation. We observe that $1\rho m$ but $3\bar{\rho}n$. Again if we define a relation R as, for $a,b \in A$ if and only if $a + b$ is even then $R = \{(1,1),(1,3)(2,2),(3.1),(3.3)\}$

Here R is a relation on the set A.

We also observe that 1 R 3 but $1\bar{R}2$.

It is to be noted that for a set A, since any subset of A×A is defined to be a relation on A hence A×A itself and the null set ϕ itself to be relations on A. They are called the universal relation and the empty relation respectively.

Let ρ be a relation from a set A to a set B. Then the domain of ρ, denoted by

$D(\rho)$ and is defined as $D(\rho) = \{a : a \in A$ and there exists $b \in B$ such that $(a,b) \in \rho\}$

Again the image of ρ, denoted by Im(ρ) and is denoted as

$\text{Im}(\rho) = \{b : b \in B \text{ and there exists } a \in A \text{ such that } (a,b) \in \rho\}$

Again the codomain of ρ is the whole set B. Thus range of ρ ⊆ codomain of ρ.

Let ρ be any relation from a set A to a set B. the inverse of the relation ρ denoted by ρ^{-1} is a relation from the set B to the set A and is defined by $\rho^{-1} = \{(b,a) : (a,b) \in \rho\}$

For example the inverse of the relation $\rho = \{(1,m),(2,n)\}$ from $A=\{1, 2, 3\}$ to $B = \{l, m, n\}$ is a relation $\rho^{-1} = \{(m,1),(n,2)\}$ from the set B to the set A.

Clearly for any relation $\rho, (\rho^{-1})^{-1} = \rho$.

Also the domain and range of ρ^{-1} are equal to the range and domain of ρ respectively.

5.2.2 Representations of relations :

A relation is represented either by Roster method or by set-builder method.

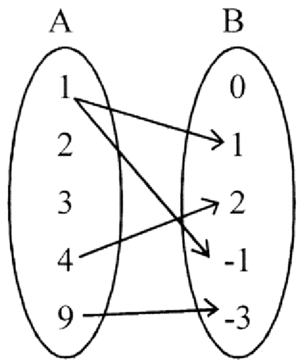
Consider an example of two sets $A = \{1,2,3,4,9\}$ and $B = \{0,1,2,-1,-3\}$

Let the relation ρ from A to B is such that $a \; \rho \; b$ holds if and only if $a \in A$ and $b \in B$ and $a = b^2$

In set-builder from we write $\rho = \{(a,b) : a \in A \text{ and } b \in B \text{ such that } a = b^2\}$

In roster form we write $\rho = \{(1,1), (1,-1), (4,2), (9,-3)\}$

We can also represent the above relations pictorially by arrow diagram as follows :



We can also represent the above relation by a rectangular array whose rows are lebeled by the elements of A and whose columns are lebeled by the elements of B.

We put 1 or 0 in each position of the arry according as for $a \in A$ and $b \in B$, $a\,\rho\,b$ holds or $a\,\rho\,b$ does not hold respectively. This array is called the matrix array of the relation.

Thus the matrix array of the above relation is as follows :

A \ B	0	1	2	−1	−3
1	0	1	0	1	0
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	1	0	0
9	0	0	0	0	1

Again if we consider a relation ρ on the set of real numbers R then as $\rho \subseteq R \times R = R^2$ and as R^2 can be represented by the set of points in a plane hence we can draw the picture of ρ by plotting those points in the plane which belong to ρ. This pictorial representation is sometimes called the graph of the relation.

5.3 Composition of relations

Let ρ be a relation from a set A into a set B and σ be relation from B into C.

Then composite relation $(\sigma_0\,\rho)$ is the relation from A to C defined by

$\sigma_0\rho = \{(a, c) : a \in A, c \in C$ and there exists some element $b \in B$ such that $(a,b) \in \rho$ and $(b,c) \in \sigma\}$

Thus $D(\sigma_0\,\rho)$ is a subset of $D(\rho)$ and $\text{Im}(\sigma_0\,\rho)$ is subset of $\text{Im}(\sigma)$. Also $(\sigma_0\,\rho)$ is a subset of $A \times C$.

By our definition $a(\sigma_0\,\rho)c$ holds if there exists some $b \in B$ such that $a\,\rho\,b$ and $b\,\sigma\,c$ for $a \in A, c \in C$.

As for example if $A = \{1,2,3\}$, $B = \{5,10,15,20\}$, $C = \{p,q,r,s\}$ and ρ is defined by $\rho = \{(1,5),(1,20),(2,10),(3,10),(4,20)\}$

and σ is defined by

$\sigma = \{(5,q),(5,r),(10,p),(15,s),(20,r)\}$

then $\sigma_0 \rho = \{(1,q),(1,r),(2,p),(3,p),(4,r)\}$

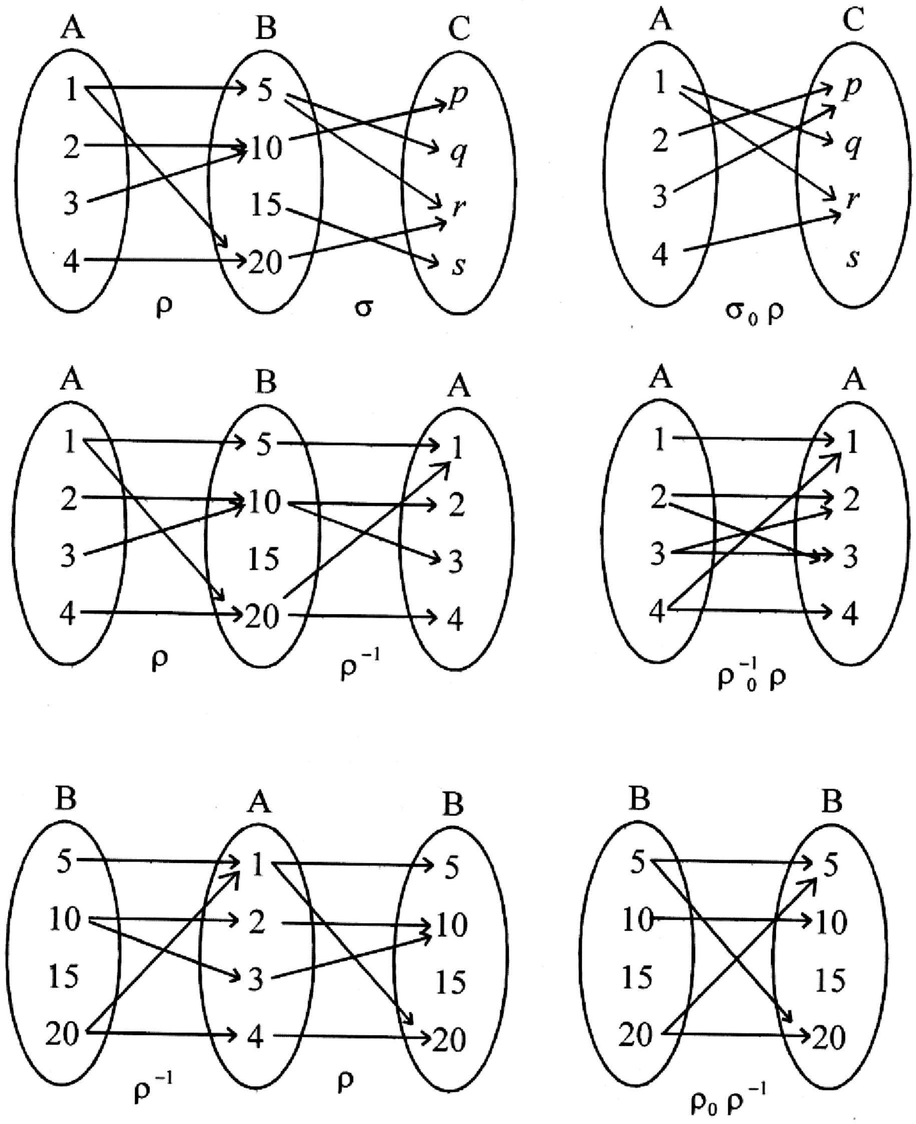
$\rho^{-1} = \{(5,1),(20,0),(10,2),(10,3),(20,4)\}$

$\rho_0^{-1} \rho = \{(1,1),(2,2),(2,3),(3,2),(3,3),(4,1),(4,4)\}$

$\rho_0 \rho^{-1} = \{(5,5),(5,20),(20,5),(20,20),(10,10)\}$

Thus in general $\rho_0^{-1} \rho \neq \rho_0 \rho^{-1}$.

The above results can be pictorially explained as follows :



5.4 Types of relations

Let A be a nonempty set and ρ be a relation on A. Then ρ is called

(i) reflexive if for all $a \in A$, $a \, \rho \, a$ holds.

(ii) symmetric if for $a, b \in A$, whenever $a \, \rho \, b$ holds, $b \, \rho \, a$ must hold.

(iii) transitive, if for $a, b, c \in A$, whenever $a \, \rho \, b$ and $b \, \rho \, c$ hold, $a \, \rho \, c$ must hold.

Thus a relation ρ is not reflexive if there exists $a \in A$ such that $(a, a) \notin \rho$ i.e., $a \, \rho \, a$ does not hold. Similarly a relation ρ is not symmetric if there exist $a, b \in A$ such that $(a, b) \in \rho$ but $(b, a) \notin \rho$. Also a relation is not transitive if there exist $a, b, c \in A$ such that $(a, b) \in \rho$ and $(b, c) \in \rho$ but $(a, c) \notin \rho$.

We have already learnt in 5.1.1 that a relation ρ on set A is called void relation or empty relation if no element of set A is related to any element of A. Again a relation ρ on set A is called universal relation if every element of the set A is related to every element of the set A.

For example let us consider four relations $\rho_1, \rho_2, \rho_3, \rho_4$ defined on the set Z where

$a \rho_1 b \Leftrightarrow a, b \in Z$ and $a \le b$

$a \, p_2 b \Leftrightarrow a, b \in Z$ and $ab \ge 0$

$a \rho_3 b \Leftrightarrow a, b \in Z$ and $a - b$ is divisible by 3

$a \rho_4 b \Leftrightarrow a, b \in Z$ and $a.b > 0$

We observe that ρ_1 is reflexive since $a \le a \; \forall \; a \in Z$

Again ρ_1 is transitive since $a \le b$ and $b \le c \Rightarrow a \le c$, for $a, b, c \in Z$

But ρ_1 is not symmetric since for $a, b \in Z$, $a \le b$ does not necessarily implies $b \le a$

Again we see that ρ_2 is reflexive, symmetric but not transitive. Because

$5 \, \rho_2 0$ and $0 \, \rho_2 - 7$ hold $(\because 5.0 \ge 0 \text{ and } 0.(-7) \ge 0)$

but $5 \, \rho_2 (-7)$ does not hold $(\because 5.(-7) = -35 < 0)$

This means for $a, b, c \in Z$, $ab \ge 0$ and $bc \ge 0$, does not necessarily implies $ac \ge 0$

It can be verified that ρ_3 is reflexive, symmetric and transitive as follows :

ρ_3 is reflexive since for any $a \in Z$, $a - a$ is divisible by 3.

ρ_3 is symmetric since for $a, b \in Z$ if $(a, b) \in \rho_3 \Rightarrow a - b$ is divisible by 3

$\Rightarrow b - a$ is divisible also by 3

ρ_3 is transitive, since for $a, b, c \in Z$ if $a - b$ is divisible by 3 and $b - c$ is divisible by 3 then $(a - b) + (b - c) = a - c$ is also divisible by 3.

In a similar manner it can be shown that ρ_4 is symmetric, transitive but not reflexive.

5.4.1 Equivalence relation

A relation ρ on a set A is called an equivalence relation if it is reflexive, symmetric and transitive.

We have seen in article 5.3 that the relation ρ_3 defined on the set Z by

"$a \ \rho_3 \ b$ if and only if $a - b$ is divisible by 3" for $a, b \in Z$ is an equivalence relation.

It is very simple to observe that the relation ρ defined on R (the set of real numbers) by "$a \ \rho \ b$ if and only if $a = b$" $a, b \in R$ is also an equivalence relation. But ρ_1 and ρ_2 as defined in article 5.3 are not equivalence relations.

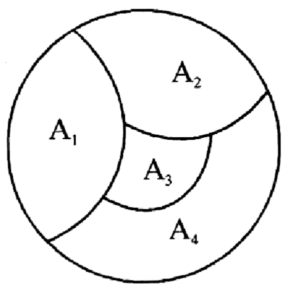
5.5 Partition

Let A be a nonempty set and P be collection of nonempty subsets of A that is $P = \{A_i : i \in I\}$, I being the index set and $A_i \subseteq A$. Now P is called a partition of A if

(i) $\bigcup_{i \in I} A_i = A$

(ii) $A_i \cap A_j = \phi$ for $i, j \in I$ and $i \neq j$

The Venn diagram of partition of a set A into four nonempty subsets A_1, A_2, A_3, A_4 (say) can be defined as follows :



For example let A, B, C be three subsets of N defined by

$A = \{3n : n \in N\} = \{3, 6, 9, 12, ...\}$

$B = \{3n - 2 : n \in N\} = \{1, 4, 7, 10.....\}$

and $C = \{3n - 1 : n \in N\} = \{2, 5, 8, 11, ...\}$

Clearly A, B, C are such that each is nonempty

and $A \cup B \cup C = N$ and $A \cap B = B \cap C = C \cap A = \phi$

Thus the subsets A, B, C form a partition of the set N.

5.6 Relation of congruence

Let m be a positive integer. For integers a, b i.e. $a, b \in Z$ we say that a is congruent to b modulo m and write $a \equiv b$ (mod m) provided $a - b$ is divisible by m.

Equivalently $a \equiv b$ (mod m) holds if and only $a, b \in Z$ and $a - b$ is of the form of $a - b = k\,m$ for some integer $k \in Z$.

Now we shall prove that the relation of congruence modulo m is an equivalence relation on Z.

Proof :

For any $a \in Z$, $a - a$ is divisible by m hence $a \equiv a$ (mod m) for any $a \in Z$.

Thus the relation of congruence (mod m) is reflexive.

Again for $a, b \in Z$ if $a \equiv b$ (mod m) holds then $a - b$ is divisible by m and hence $b - a$ is also divisible by m. Hence the relation of congruence (mod m) is symmetric.

Finally for any $a, b, c \in Z$ if $a \equiv b$ (mod m) and $b \equiv c$ (mod m) then as $a - b$ and $b - c$ is divisible by m hence $(a - b) + (b - c)$ i.e. $a - c$ is also divisible by m.

Hence the relation of congruence (mod m) is transitive.

Hence the relation of congruence (mod m) on the set Z is an equivalence relation.

5.7 Equivalence class

Let ρ be an equivalence relation on a set S.

For each element $a \in S$, *let* $[a]$ or $cl(a)$ defined as $[a]$ or $cl(a)$ $= \{x \in S : x\,\rho\,a\}$

This set $[a]$ or $cl(a)$ is called the ρ-equivalence class of a in S.

Thus $cl(a)$ or $[a]$ is the set of those elements x of S such that $x \rho a$ holds.

For example consider the relation ρ defined on Z by " $a \rho b$ if and only if $a - b$ is divisible by 3" for $a, b \in Z$. We have seen earlier that it is an an equivalence relation of congruence (mod 3).

Now $cl(0)$ or $[0] = \{x \in Z : x \rho 0\}$

$= \{x \in Z : x - 0 \text{ is divisible by } 3\} = \{0, \pm 3, \pm 6, ...\}$

$cl(4) = \{x \in Z : x \rho 4\}$

$= \{x \in Z : x - 4 \text{ is divisible by } 3\}$

$= \{........, -5, -2, 1, 4, 7, 10......\}$

$= \{1, 1 \pm 3, 1 \pm 6, ...\}$

$cl(1) = \{x \in Z : x \rho 1\}$

$= \{x \in Z : x - 1 \text{ is divisible by } 3\}$

$= \{1, 1 \pm 3, 1 \pm 6, ...\}$ etc.

The collection of all equivalence classes of elements of a set S under an equivalence relation ρ is denoted by S/ρ and is defined as $S/\rho = \{[a]/a \in S\}$

It is called the quotient set of S by ρ.

The fundamental property of quotient set S/ρ is contained in the following theorem.

Theorem : Let ρ be an equivalence relation of a set S and $a, b \in S$. Then

(i) For each a in S, $a \in cl(a)$

(ii) $cl(a) = cl(b) \Leftrightarrow (a, b) \in \rho$

(iii) If $a \rho b$ does not hold then $cl(a)$ and $cl(b)$ are disjoint.

(iv) Two classes $cl(a)$ and $cl(b)$ are either disjoint or equal.

(v) The quotient set S/ρ is a partition of S.

(vi) Each partition of S yields an equivalence relation.

Proof

(i) Since ρ is an equivalence relation, hence ρ is reflexive and hence for any $a \in S$, $a\ \rho\ a$ holds. Thus $a \in cl(a)$

(ii) Let $cl(a)=cl(b)$ and let $x \in cl(a)$ $\therefore x \in cl(b)$

Since $x \in cl(a)$ hence $x\rho a$ holds.

Again since $x \in cl(b)$ hence $x\rho b$ hods.

Now $x\ \rho\ a$ *and* $x\ \rho\ b$.

$\Rightarrow a\ \rho\ x$ and $x\ \rho\ b$ ($\because \rho$ is symmetric)

$\Rightarrow a\ \rho\ b$ ($\because \rho$ is transitive)

$\therefore cl(a) = cl(b) \Rightarrow a\ \rho\ b$

Conversely let $a\ \rho\ b$ and let $x \in cl(a)$

$\therefore$ $x\ \rho\ a$ holds. Hence $x\ \rho\ a$ and $a\ \rho\ b$

$\Rightarrow x\ \rho\ b$ ($\because \rho$ is transitive)

$\Rightarrow x \in cl(b)$

Thus if $a\ \rho\ b$ and $x \in cl(a) \Rightarrow x \in cl(b)$

Thus if $a\ \rho\ b$ then $cl(a) \subseteq cl(b)$

Similarly it can also be shown that if $a\ \rho\ b$ and $x \in cl(b) \Rightarrow x \in cl(a)$

$\Rightarrow cl(b) \subseteq cl(a)$

Hence $a\ \rho\ b \Rightarrow cl(a) \subseteq cl(b)$ and $cl(b) \subseteq cl(a)$

Thus it follows that $a\ \rho\ b \Rightarrow cl(a) = cl(b)$

Hence $cl(a) = cl(b) \Leftrightarrow a\ \rho\ b$

(iii) It is given that $a\ \rho\ b$ does not hold.

We have to prove that $cl(a) \cap cl(b) = \phi$.

If possible let $cl(a) \cap cl(b) \neq \phi$ and thus let $x \in cl(a) \cap cl(b)$

$\therefore x\ \rho\ a$ and $x\ \rho\ b$

$\Rightarrow a \rho x$ and $x \rho b$ ($\because \rho$ is symmetric)

$\Rightarrow a \rho b$ ($\because \rho$ is transitive)

This is a contradiction. Hence $cl(a) \cap cl(b) = \phi$

(iv) Since $a, b \in S$ two cases my arise, either $a \rho b$ holds or $a \rho b$ does not hold.

Now if $a \rho b$ hold then by the proof of part (ii) we have $cl(a) = cl(b)$

Again if $a \rho b$ does not hold then by the proof of part (iii) we have $cl(a) \cap cl(b) = \phi$

Thus either $cl(a) = cl(b)$ or $cl(a) \cap cl(b) = \phi$

(v) We know that the quotient set S/ρ is the family of distinct ρ-equivalence class. Again we have already proved that for any $a, b \in S$ the ρ-equivalence class $cl(a)$, $cl(b)$ are either disjoint or equal. Thus in the quotient set S/ρ,

(a) each class in S/ρ is nonempty (since for any $cl(a) \in S/\rho$, $a \in cl(a)$)

(b) the union of family of classes is the set S that is $\bigcup_{a \in S/\rho} cl(a) = S$ and

(c) the classes are pairwise distinct.

Therefore the family of distinct ρ-equivalence classes that is the quotient set S/ρ is a partition of S.

(vi) Let there be a partition P of the set S into subsets.

Let us define a relation ρ on the set S as follows :

"$a \rho b$ holds if and only if a and b belong to one and the same subset of the partition P" for $a, b \in S$.

Now for any $a \in S$, $a \rho a$ holds since a and a belong to one and the same subset of the parition P. Therefore ρ is reflexive.

Again let $a, b \in S$, such that $a \rho b$ holds. Then a and b belong to one and the same subset of the same subset of the partition P and therefore b and a belong to the same subset of the partition P. Therefore $b \rho a$ holds.

Thus $a \rho b \Rightarrow b \rho a$. Hence ρ is symmetric.

Again let $a, b, c \in S$ and $a \rho b$, $b \rho c$ both bold.

Then a and b belong to one and the same subset $S_1(say)$ of P i.e. $S_1 \in P$.

Also b and c belong to one and the same subset $S_2(say)$ of P i.e. $S_2 \in P$.

Thus $b \in S_1$ and $b \in S_2$ hence $b \in S_1 \cap S_2$. Thus $S_1 \cap S_2 \neq \phi$.

Again since P is a partition and $S_1, S_2 \in P$ hence S_1 and S_2 must be either disjoint or equal.

Again since $S_1 \cap S_2 \neq \phi$ hence $S_1 = S_2$. Thus a and c belong to one and the same subset $S_1(\equiv S_2)$. Therefore ρ is transitive.

Thus ρ is an equivalence relation. This completes the proof.

Let us verify the above theorem with some examples.

Let us consider the relation ρ of congruence (mod 3) defined on Z.

We have seen that this relation ρ is an equivalence relation. Now we observe that

$$cl(0) = \{0, \pm 3, \pm 6, ...\} = \{3n / n \in Z\} = cl(3) = cl(-3) = cl(6) = cl(-6) \text{ etc.}$$

$$cl(1) = \{1+0,\ 1 \pm 3,\ 1 \pm 6, ...\} = \{3n+1 / n \in Z\} = cl(4) = cl(-2) = cl(7) = cl(-5) \text{ etc.}$$

$$cl(2) = \{2+0, 2 \pm 3, 2 \pm 6, ...\} = \{3n+2 / n \in Z\} = cl(5) = cl(-1) = cl(8) = cl(-4) \text{ etc.}$$

Thus there are three distinct ρ-equivalence classes and any two classes are either disjoint or equal. Thus the quotient set $Z/\rho = \{cl(0), cl(1), cl(2)\}$ is a partition of Z consists of three distinct ρ-equivalence classes.

Thus the results of the given theorem is verified.

More over consider the partition $P = \{E, O\}$ of Z where E is the set of even integers and O is the set of odd integers then this partition yields the equivalence relation as follows :

"a ρ b holds if and only if $a - b$ is divisible by 2" for $a, b \in Z$.

Hence for every partition of a set we can yield an equivalence relation.

5.8 Antisymmetric relation

We know that a relation ρ on a nonempty set S is symmetric if $a\,\rho\,b \Rightarrow b\,\rho\,a$ for $a, b \in S$. Thus ρ is not symmetric if there exists $a,b \in S$ such that $(a,b) \in \rho$ but $(b,a) \notin \rho$. We now discuss an another kind of relation known as antisymmetric relation.

A relation ρ on a nonempty set S is said to be antisymmetric if for $a,b \in S$, whenever both $a\rho b$ and $b\rho a$ hold then $a = b$. This means whenever $(a,b), (b,a) \in \rho$ then $a = b$. Thus ρ is not antysymmetric if there exists $a,b \in S$ such that (a, b) and $(b,a) \in \rho$ but $a \neq b$.

For example consider the two relations ρ_1 and ρ_2 on the set $S = \{1,2,3\}$ where $\rho_1 = \{(1,1),(1,2),(2,1)\}$, $\rho_2 = \{(1,3),(2,1)\}$

Now ρ_1 is not antisymmetric since $(1,2) \in \rho_1$, $(2,1) \in \rho_1$ but $1 \neq 2$.

Again ρ_2 is antisymmetric since we cannot find any $a,b \in S$ such that both $a\,\rho\,b$ and $b\,\rho\,a$ hold.

Again let us consider the relation ρ_3 defined on the set Z as follows :

"$a\ \rho_3\ b$ hold if and only if $a \leq b$" for $a,b \in Z$.

By our discussion it is clear that if ρ_4 and ρ_5 are empty relation and universal relations respectively on a set $S = \{1,2,3\}$ then ρ_4 is antisymmetric but ρ_5 is not antisymmetric.

5.9 Partial ordering relation

Now we define an another important class of relations known as partial order relation. A relation ρ on a nonempty set S is said to be partial ordering or partial order of S if ρ is reflexive, antisymmetric and transitive. A partial order relation is often denoted by '$\leq$' even if it is not "less than or equal to".

A nonempty set S together with a relation of partial order defined on S is called a partial order set or poset.

For example let S be any nonempty set and $P(S)$ be the power set of S.

Now for any two subsets A, B of S that is for any $A, B \in P(S)$ either $A \subseteq B$ or $A \not\subseteq B$.

Hence '$\subseteq$' (of being subset) defines a relation of $P(S)$.

Since for any $A \in P(S)$, $A \subseteq A$ hence the relation '$\subseteq$' is reflexive.

Again for any $A, B \in P(S)$ if $A \subseteq B$ and $B \subseteq A$ then by the definition of equality of sets $A = B$. Thus the relation '$\subseteq$' is antisymmetric.

Again for any $A, B, C \in P(S)$ if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ hence the relation '$\subseteq$' is transitive. Consequently the relation '$\subseteq$' (where $A \subseteq B$ means A is a subset of B) is a partial order relation on $P(S)$ and so $(P(S), \subseteq)$ is a poset.

Let us consider an another example. It can be proved that (N, ρ) is a poset where N is the set of natural numbers and $m \rho n$ means "m is a divisor of n" for $m, n \in N$, that is $\rho = \{(m,n) \in N \times N : m \text{ is a divisor of } n\}$.

Now the relation ρ is antisymmetric. Because if $a \rho b$ and $b \rho a$ holds for some $a, b \in N$ then as a is a divisor of b hence b can be expressed as $b = la$ where $l \in N$ and as b is a divisor of a hence a can be expressed as $a = pb$ where $p \in N$

$\therefore b = la = lpb \Rightarrow lp = 1 \Rightarrow l = 1$ and $p = 1$ (since $l, p \in N$) $\Rightarrow a = b$

Hence ρ is antisymmetric. In a similar manner we can also prove that ρ is reflexive and transitive.

Hence ρ is a partial order relation on N.

In a similar manner one can also easily verify that $(R, \leq)$ is a poset where R is the set of real numbers and $x \leq y$ means "x is less than or equal to y" for $x, y \in R$.

5.10 n-ary relation

All the relations discussed in the previous articles were binary relations. In binary relation we derived a relation from ordered pair of elements. Instead if one defines a relation of the ordered n-tuple then one may talk about n-ary relation.

For example let us consider two sets A_1 of students and A_2 of subjects as follows :

A_1 = {Amal, Bimal, Chhanda}

A_2 = {Physics, Mathematics, Computer Science}

Let us consider a relation ρ involving the roles of the students of A_1 that might play with two subjects of A_2 which can be expressed as a statement of the form :

ρ : "A student $x \in A_1$ thinks that a subject $y \in A_2$ is more interesting than a subject $z \in A_2$".

Also we assume that the above statement is true for the following values of x,y,z in tabular form

Statement (x)	Subject (y)	Subject (z)
Amal	Physics	Mathematics
Amal	Mathematics	Computer Science
Bimal	Mathematics	Physics
Chhanda	Computer Science	Mathematics
Chhanda	Mathematics	Physics

Each row of the table records a fact that "x thinks that y is more interesting than z".

For instance from the first row we have an information that "Amal thinks that Physics is more interesting than Mathematics".

Thus the table represents the relation ρ among a student $x \in A_1$, a subject $y \in A_2$ and subject $z \in A_2$. Thus the data of the table are equivalent to the following set ordered trip-lets.

ρ = {(Amal, Physics, Mathematics), (Amal, Mathematics, Computer Science), (Bimal, Mathematicds, Physics), (Chhanda, Computer Science, Mathematics), (Chhanda, Mathematics, Physics)}

The relation ρ is a ternary relation since there are three items involve in each row. Obviously this relation is a subset of the cartesian product $A_1 \times A_2 \times A_2$.

By generalising this concept, we can define ordered n tuple, cartesian product of n sets and n-ary relation.

An ordered n tuple is a set of n objects with an ordered associated with them. If n objects are represented by $x_1, x_2 \ldots\ldots\ldots\ldots x_n$ then we write the ordered n tuple as $(x_1, x_2 \ldots\ldots\ldots\ldots x_n)$. Let us consider n sets $A_1, A_2 \ldots\ldots\ldots\ldots A_n$. Then the set of all ordered n-tuples $(a_1, a_2 \ldots\ldots\ldots\ldots a_n)$ where $a_i \in A_i$, $1 \leq i \leq n$ is called the cartesian product of sets $A_1, A_2 \ldots\ldots\ldots\ldots A_n$ and is denoted by $A_1 \times A_2 \times \ldots\ldots\ldots\ldots \times A_n$.

Let us now define an n-ary relation as follows :

An n-ary relation on sets A_1, A_2 .... A_n is a set of ordered n-tuples (a_1, a_2 .... a_n) where $a_i \in A_i$ for all $i = 1, 2.......n$.

Thus n-ary relation on sets A_1, A_2 .... A_n is a subset of cartesian product $A_1 \times A_2 \times ....\times A_n$.

In particular if $A_1 = A_2 = ... = A_n = A$ (*say*) then for a set A, an n-ary relation is a subset of the cartesian product $A \times A \times .... \times A$ or A^n.

Now if we take $n = 3$ then a subset of A^3 is called a ternary relation on A.

For example let A_1, A_2, A_3 be the set of names, addresses and mobile numbers as follows :

A_1 = {Amal, Bina, Osman}

A_2 = {25 Ghoshpara Street, 51 Central Avenue, 10 Mahatma Gandhi Road}

A_3 = {9547212379, 8237178234, 8723456213}

Then the set of 3-tuples {(Amal, 51 Central Avenue, 9547212379), (Osman, 25 Ghoshpara Street, 8723456213)} is a 3-ary relation over A_1, A_2 and A_3, since the above set is the subset of the cartesian product $A_1 \times A_2 \times A_3$.

For an another example consider the subset

$\rho = \{(a_1, a_2, ...a_n) \in Z^n : a_1, a_2, ...a_n$ are in A.P.$\}$ of the set Z^n.

This ρ is an n-ary relation on Z (the set of integers).

Clearly $(1,3,5,........, 2n-1) \in \rho$ but $(1^2, 2^2, 3^2, ...n^2) \notin \rho$

5.11 Examples

5.11.1 Choose the correct option

(i) Let A and B be two sets such that n(A) = m and $n(B) = p$. Then the total number of nonempty relations that can be defined from A to B is—

(a) m^p (b) p^m-1 (c) $mp - 1$ (d) $2^{mp}-1$

(ii) Let R be a relation on N defined by $x + 2y = 8$.

The domain of R is (a) {2,4,8} (b) {2,4,6,8} (c) {2,4,6} (d) {1,2,3,4}

(iii) Let R be the relation from A = {2,3,4,5} to B = {3,6,7,10} defined by "x divides y", for $x \in A$ and $y \in B$. Then R^{-1} is equal to

(a) {(6,2), (3,3)}, (b) {(6,2), (10,2), (3,3), (6,3), (10,5)}

(c) {(6,2), (10,2), (3,3), (6,3)} (d) none of the these

Solution

(i) Given $n(A) = m,\ n(B) = p\ \therefore (A \times B) = m.p$

Total no. of relations from A to B = 2^{mp}

Total no. of nonempty relations from A to B = 2^{mp-1}. Therefore the correct option is (d)

(ii) Given $R = \{(x, y) : x + 2y = 8,\ x, y \in N\}$

Now $x + 2y = 8 \Rightarrow y = \dfrac{8-x}{2}$

$\therefore R = \{(2,3), (4,2), (6,1)\}$

$\therefore$ Domain of R = {2,4,6}. Therefore the correct option is (c)

(iii) Given A = {2,3,4,5}, B = {3,6,7,10}

$R = \{(x, y) : x \text{ divides } y,\ x \in A,\ y \in B\} = \{(2,6), (2,10), (3,3), (3,6), (5,10)\}$

$\therefore R^{-1} = \{(6,2), (10,2), (3,3), (6,3), (10,5)\}$

Hence the correct option is (b).

5.11.2 Prove that the inverse of an equivalence relation is an equivalence relation.

Solution

Let ρ be an equivalence relation on a set S. Then ρ is reflexive, symmetric and transitive.

(i) Since ρ is reflexive hence $(a,a) \in \rho,\ \forall\ a \in S$.

Thus $(a,a) \in \rho^{-1}\ \forall a \in S$. Hence ρ^{-1} is reflexive.

(ii) Let $(a,b) \in \rho^{-1}$ for some $a, b \in S$

$\therefore (b,a) \in \rho \Rightarrow (a,b) \in \rho$ ($\because$ ρ is symmetric)

$\Rightarrow (b,a) \in \rho^{-1}$

Thus $(a,b) \in \rho^{-1} \Rightarrow (b,a) \in \rho^{-1}$. Hence ρ^{-1} is symmetric.

(iii) Let for some $a,b,c \in S$, $(a,b) \in \rho^{-1}$ and $(b,c) \in \rho^{-1}$

Then $(b,a) \in \rho$ and $(c,b) \in \rho$.

Since $(c,b) \in \rho$ and $(b,a) \in \rho$ and ρ is transitive hence $(c,a) \in \rho \Rightarrow (a,c) \in \rho^{-1}$

Thus $(a,b) \in \rho^{-1}$ and $(b,c) \in \rho^{-1} \Rightarrow (a,c) \in \rho^{-1}$

Therefore ρ^{-1} is transitive.

Hence ρ^{-1} is also an equivalence relation.

5.11.3 Let ρ be relation on a set A. Prove that ρ is transitive if and only if $\rho_0\rho \subseteq \rho$.

Solution

Let ρ be the transitive. We shall prove that $\rho_0\rho \subseteq \rho$.

Let $(a,c) \in \rho_0\rho$. This means there exists some $b \in A$ such that $(a,b) \in \rho$ and $(b,c) \in \rho$

Since ρ is transitive and (a, b), $(b, c) \in \rho$ hence $(a,c) \in \rho$

Thus $(a,c) \in \rho_0\rho \Rightarrow (a,c) \in \rho \Rightarrow \rho_0\rho \subseteq \rho$

Conversely let $\rho_0\rho \subseteq \rho$

Let for some a,b,c, $(a,b) \in \rho$ and $(b,c) \in \rho$

Then since $\rho_0\rho \subseteq \rho$ hence $(a,c) \in \rho$

Thus $(a,b) \in \rho$ and $(b,c) \in \rho \Rightarrow (a,c) \in \rho$

Hence ρ is transitive.

This completes the proof.

5.11.4 For each of the following relations on $A = \{1,2,3\}$ decide whether or not each of the following relation on A is reflexive, symmetric, antisymmetric or transitive. Also identify the equivalence relations.

(i) $\rho = \{(1,3), (3,1)\}$

(ii) $\rho = \{(1,1)\}$

(iii) $\rho = \{(1,2), (1,3)\}$

(iv) $\rho = \{(1,1), (2,2), (3,3), (2,3), (3,2)\}$

(v) $\rho = \{(1,1), (2,2), (3,3), (2,3)\}$

(vi) $\rho = \phi$ (empty relation)

(vii) $\rho = A \times A$ (universal relation)

Solution

(i) Since $(1,1) \notin \rho$, ρ is not reflexive

ρ is symmetric since whenever $(a,b) \in \rho$ we also have $(b,a) \in \rho$.

ρ is not antisymmetric since $(1,3) \in \rho$ and $(3,1) \in \rho$ but $1 \neq 3$

ρ is not transtive since $(1,3) \in \rho$ and $(3,1) \in \rho$ but $(1,1) \notin \rho$

(ii) Here ρ is not reflexive since $(2,2) \notin \rho$

ρ is not symmetric whenever $(a,b) \in \rho$ we also have $(b,a) \in \rho$

By a similar argument it can be shown that $(b,a) \in \rho$ is antisymmetric and transitive.

(iii) ρ is not reflexive since $(1,1) \notin \rho$

ρ is not symmetric since $(1,2) \in \rho$ but $(2,1) \notin \rho$

ρ is transitive as there do not exist any two pair of the form (a,b), (b,c) in ρ

ρ is antisymmetric as there do not exist any two pair of the form (a,b) and (b,a) in ρ

(iv) It is easy to verify that ρ is reflexive symmetric and transitive but not antisymmetric $(\text{Since } (2,3), (3,2) \in \rho \text{ but } 2 \neq 3)$. Since ρ is reflexive symmetric and transitive, hence it is an equivalence relation.

(v) Clearly ρ is reflexive, transitive and antisymmetric but it is not symmetric since $(2,3) \in \rho$ but $(3,2) \notin \rho$.

(vi) The empty relation ϕ is not reflexive since $(1,1) \notin \rho (= \phi)$

ρ is symmetric, antisymmetric and transitive as there is no element (a,b) in ρ.

(vii) $\rho = A \times A$ is obviously reflexive, symmetic and transitive but it is not antisymmetric since $(1,2) \in \rho$ and $(2,1) \in \rho$ but $1 \neq 2$. Since ρ is reflexive symmetric and transitive, hence it is an equivalence relation.

5.11.5 Let R and S be two relation on A = {1,2,3} where

$R = \{(1,2),(1,3),(2,1)(3,3)\}$ and $S = \{(1,1),(1,2),(2,3),(3,1),(3,3)\}$

Find (i) $R \cap S$ (ii) $R \cup S$ (iii) $\bar{S}$ (iv) RoS (v) $R^2 = RoR$

Solution

Here R = {(1,2), (1,3), (2,1), (3,3)} and S={(1,1), (1,2), (2,3), (3,1), (3,3)}

Also R and S are subsets of universal set $A \times A$.

(i) $R \cap S$ = the usual intersection of two sets R and S = {(1,2), (3,3)}

(ii) $R \cup S$ = the usual union of two sets R and S = {(1,1), (1,2), (1,3), (2,1), (2,3), (3, 1), (3,3)}

(iii) $\bar{S}$ = the usual complement of $S = A \times A - S = \{(1,3),(2,1),(2,2),(3,2)\}$

(iv) We know that $R \circ S = \{(a,c) : a \in A,\ c \in A$ and there exists some element $b \in A$ such that $(a,b) \in S$ and $(b,c) \in R\}$

Thus as $(1,1) \in S$ and $(1,2), (1,3) \in R$ hence $(1,2),(1,3) \in R \circ S$

Similarly since $(1,2) \in S$ and $(2,1) \in$ R hence $(1,1) \in R \circ S$

Also since $(2,3) \in S$ and $(3,3) \in R$ hence $(2,3) \in R \circ S$

Similarly since $(3,1) \in S$ and $(1,2), (1,3) \in R$ hence $(3,2),(3,3) \in R \circ S$

Again as $(3,3) \in S$ and $(3,3) \in R$ hence $(3,3) \in R \circ S$

Therefore $R \circ S = \{(1,2),(1,3),(1,1),(2,3),(3,2),(3,3)\}$

(v) Following the arguments in (iv) $R \circ R = \{(1,1),(1,3),(2,2),(2,3),(3,3)\}$

5.11.6 Examine if the relation ρ on the set S is—(i) reflexive (ii) symmetric (iii) antisymmetric (iv) transitive. Also identify the equivalence relations.

(a) S = Z and ρ is defined on Z by "a ρ b if and only if $2a + 3b$ is divisible by 5" for $a,b \in Z$

(b) S = $N \times N$ and ρ is defined on $N \times N$ by "(a, b) ρ (c, d) if and only if $ad(b+c) = bc(a+d)$" for $(a,b),(c,d) \in N \times N$

(c) $S = Z$ and ρ is defined on Z by "$a \rho b$ if and only if $a - b < 4$" for $a, b \in Z$

(d) S is the set of all straight lines in a plane and ρ is defined by "$l \rho m$ if and only if l is perpendicular to m" for $l, m \in S$.

Solution

(a) (i) ρ is reflexive since for any $a \in Z$, $a\rho b$ holds that is $2a + 3a$ is divisible by 5.

(ii) If for some $a, b \in Z$, $a\rho b$ holds that is $2a + 3b$ is divisible by 5 then $5(a + b) - (2a + 3b)$ or $3a + 2b$ is also divisible by 5. This means $a \rho b \Rightarrow b \rho a$. Hence ρ is symmetric.

(iii) $(1,6) \in \rho$ since $2.1+3.6=20$ is divisible by 5. Also $(6,1) \in \rho$ since $2.6+3.1=15$ is also divisible by 5.

Thus (1,6) and $(6,1) \in \rho$ but $1 \neq 6$ hence ρ is not antisymmetric.

(iv) If for some $a, b, c \in Z$, $a\rho b$ and $b\rho c$ hold then $(2a + 3b)$ and $(2b + 3c)$ is divisible by 5. Thus $(2a + 3b) + (2b + 3c) = (2a + 3c) + 5b$ is divisible by 5. Hence $2a + 3c$ is divisible by 5. ($\because 5b$ is divisible by 5)

Thus for $a, b, c \in Z$, $a\rho b$ and $b\rho c \Rightarrow a\rho c$. Thus ρ is transitive.

Since ρ is reflexive symmetric and transitive, hence it is an equivalence relation.

(b) Given $(a,b)\rho(c,d) \Leftrightarrow ad(b+c) = bc(a+d)$ for $(a,b),(c,d) \in N \times N$

This means $(a,b)\rho(c,d) \Leftrightarrow \frac{1}{b}+\frac{1}{c}=\frac{1}{a}+\frac{1}{d}$ for $(a,b),(c,d) \in N \times N$ since $abcd \neq 0$

(i) For any $(a,b) \in N \times N$

$(a,b)\rho(a,b)$ holds since $\frac{1}{b}+\frac{1}{a}=\frac{1}{a}+\frac{1}{b}$. Hence ρ is reflexive.

(ii) If for some $(a,b),(c,d) \in N \times N$

$(a,b)\rho(c,d)$ holds then $\frac{1}{b}+\frac{1}{c}=\frac{1}{a}+\frac{1}{d} \Rightarrow \frac{1}{d}+\frac{1}{a}=\frac{1}{c}+\frac{1}{b} \Rightarrow (c,d)\rho(a,b)$ holds.

Hence ρ is symmetric.

(iii) We see that for $a = 2$, $b = 2$, $c = 3$, $d = 3$

$(a,b),(c,d) \in N \times N$ such that both $(a,b)\rho(c,d)$ and $(c,d)\rho(a,b)$ hold

$\left(\text{since } \frac{1}{b}+\frac{1}{c}=\frac{1}{a}+\frac{1}{d}\right)$

But $(a,b) \neq (c,d)$ i.e. $(2,2) \neq (3,3)$ hence ρ is not antisymmetric.

(iv) If for some $(a,b), (c,d), (e,f) \in N \times N$

such that $(a,b)\rho(c,d)$ and $(c,d)\rho(e,f)$ hold them

$\frac{1}{b}+\frac{1}{c}=\frac{1}{a}+\frac{1}{d}$ and $\frac{1}{d}+\frac{1}{e}=\frac{1}{c}+\frac{1}{f}$

$\Rightarrow \frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}=\frac{1}{a}+\frac{1}{d}+\frac{1}{c}+\frac{1}{f} \quad \Rightarrow \frac{1}{b}+\frac{1}{e}=\frac{1}{a}+\frac{1}{f} \quad \Rightarrow (a,b)\rho(e,f)$

holds.

Hence ρ is transitive. Since ρ is reflexive symmetric and transitive, hence it is an equivalence relation.

(c) Given that $a\rho b \Leftrightarrow a-b<4$ for $a,b \in Z$

(i) For any $a \in Z$, $a-a=0<4 \Rightarrow a\ \rho\ a$ $\therefore \rho$ is reflexive.

(ii) We observe that 1 ρ 6 holds since 1 – 6 < 4 where $1,6 \in Z$.

But 6 ρ 1 does not hold since $6-1 \not< 4$. Hence ρ is not symmetric.

(iii) We observe that 2 ρ 3 and 3 ρ 2 hold since 2 – 3 < 4 and 3 – 2 < 4 where $2,3 \in Z$

But $2 \neq 3$. Hence ρ is not antisymmetric.

(iv) We observe that 6 ρ 3 and 3 ρ 1 hold since 6 – 3 < 4 and 3 – 1 < 4 where $3,1 \in Z$

But 6 ρ 1 does not hold since $6-1 \not< 4$. Hence ρ is not transitive.

(d) Given that S is the set of all lines in a plane and $l\rho m \Leftrightarrow l$ is perpendicular to m for $l,m \in S$.

(i) ρ is not reflexive since for any line $l \in S$, l is not perpendicular to itself.

(ii) For any $l,m \in S$, l is perpendicular to $m \Rightarrow m$ is perpendicular to l

$\therefore l\rho m \Rightarrow m\rho l$ Hence ρ is symmetric.

(iii) We know that two lines in a plane will be perpendicular only if they are different.

Hence $l\rho m$ and $m\rho l$ does not imply l = m. Hence ρ is not antisymmetric.

(iv) For any $l,m,n \in S$, $l\rho m$ and $m\rho n$ imply that m is perpendicular to both l and m.

This means the straight line l and n are parallel or coincident.

Thus l ρ m and m ρ n does not imply l ρ n. Thus ρ is not transitive.

5.11.7 Find the equivalence classes of ρ defined in the problems in 5.10.4 (iv) and 5.10.6 (a). Also find the corresponding partition of the set A and Z respectively induced by the relation.

Solution :

In the problem 5.10.4 (iv), ρ is defined on the set $A = \{1,2,3\}$

and $\rho = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$

We have already proved that ρ is an equivalence relation.

Now $cl(1) = \{x \in A / x\rho 1\} = \{1\}$ ($\because 1\rho 1$ holds)

$cl(2) = \{x \in A / x\rho 2\} = \{2,3\}$

$cl(3) = \{x \in A / x\rho 3\} = \{2,3\}$ ($\because$ 2 ρ 2 and 3 ρ 2 holds)

$\therefore$ There are two distinct classes for ρ and these are $cl(1)$, $cl(2)$.

Accordingly $\{cl(1), cl(2)\} = \{\{1\},\{2,3\}\}$ is the partition of A induced by ρ.

In the problem 5.10.6 (a), ρ is defined on the set Z as follows :

"a ρ b if and only if $2a + 3b$ is divisible by 5" for $a,b \in Z$

We have already seen that this relation is an equivalence relation.

Now $cl(0) = \{x \in Z / x\rho 0\}$

$= \{x \in Z / 2x + 3.0 \text{ is divisible by } 5\}$

$= \{0, \pm 5, \pm 10, ...\} = \{5n / n \in Z\}$

$cl(1) = \{x \in Z / x \rho 1\}$

$= \{x \in Z / 2x + 3.1 \text{ is divisible by } 5\}$

$= \{1, 1 \pm 5, 1 \pm 10, ...\} = \{5n + 1 / n \in Z\}$

In a similar manner, $cl(2) = \{5n + 2 / n \in Z\}, cl(3) = \{5n + 3 / n \in Z\}$

$cl(4) = \{5n + 4 / n \in Z\}$, $cl(5) = cl(0)$, $cl(6) = cl(1)$ etc.

Thus there are five distinct equivalence classes for ρ and these are

$cl(0), cl(1), cl(2), cl(3), cl(4)$

Accordingly $\{cl(0), cl(1), cl(2), cl(3), cl(4)\}$ is the partition of Z induced by ρ.

5.11.8 Let N be the set of all positive integers. Define a relation $\leq$ on N by "$a \leq b$ if and only if a is divisor of y" for $a, b \in N$. Prove that $(N, \leq)$ is a partially ordered set or poset.

Solution :

The relation $\leq$ is reflexive since for any $a \in N$, a is a divisor of a. Thus $a \leq a$ holds. Again if for some $a, b \in N$, $a \leq b$ and $b \leq a$ hold that is the statements "a is divisor of b" and "b is a divisor of a" are true then there exist same $m, n \in N$ such that

$b = ma$ and $a = nb$

$\Rightarrow b = ma = mnb \Rightarrow mn = 1$

But as $m, n \in N$ and $mn = 1$ hence $m = n = 1$.

So that we get $a = b$. Thus the relation $\leq$ is antisymmetric.

Again for any $a, b, c \in N$ if a is a divisor of b and b is a divisor of c then there exist some m and n such that $b = ma$ and $c = nb \Rightarrow c = nb = mna$

Since $m, n \in N$ hence a is a divisor of c. Thus $a \leq b$ and $b \leq c \Rightarrow a \leq c$.

Hence $\leq$ is transitive. Therefore $(N, \leq)$ is a poset.

5.12 Exercise

5.12.1 Choose the correct option

(i) Let R be a relation from $A=\{11,12,13\}$ to $B=\{8,10,12\}$ defined by $y = x - 3$ for $x \in A$, $y \in B$. The relation R^{-1} is

(a) $\{(11,8), (13,10)\}$ (b) $\{(8,11), (10,13)\}$ (c) $\{(8,11), (9,12), (10,13)\}$ (d) none of these

(ii) Let a relation ρ be defined as $\rho = \{(4,5), (1,4), (4,6), (7,6), (3,7)\}$. The relation $\rho_0^{-1}\rho$ is given by

(a) $\{(1,1), (4,4), (7,4), (4,7), (7,7)\}$ (b) $\{(1,1), (4,4), (4,7), (7,4), (7,7), (3,3)\}$ (c) $\{(1,5), (1,6), (3,6)\}$ (d) none of these

5.12.2 Let ρ be a relation on a set A. Prove that ρ is symmetric if and only if $\rho^{-1} = \rho$.

5.12.3 Give examples of relation R on A = {1,2,3} such that

(i) R is both symmetric and antisymmetric

(ii) R is neither symmetric nor antisymmetric

(iii) R is transitive but $R \cup R^{-1}$ is not transitive

5.12.4 For the partition P={{1}, {2,3}, {4,5}} write the corresponding equivalence relation on the set $A = \{1, 2, 3, 4, 5)$.

5.12.5 Let R be the set of real numbers and let $\rho_0, \rho_1, \rho_2, \ldots\ldots\ldots \rho_7$ be eight relations defined on R and for any $a, b \in R$

$a \, \rho_0 \, b \iff b = 3a$, $a \, \rho_1 \, b \iff a < b$, $a \, \rho_2 \, b \iff a \neq b$

$a \, \rho_3 \, b \iff ab > 0$, $a \, \rho_4 \, b \iff b \neq a+1$, $a \, \rho_5 \, b \iff a \leq b$

$a \, \rho_6 \, b \iff ab \geq 0$, $a \, \rho_7 \, b \iff a = b$

Test whether the relations are reflexive, symmetric and transitive.

5.12.6 Prove that a relation ρ defined on a set S is an equivalence relation if and only if ρ is reflexive and such that $a \; \rho \; b$ and $b \; \rho \; c \implies c \; \rho \; a$.

5.12.7 A relation ρ is defined on Z by " $a \; \rho \; b$ if and only if $a^2 - b^2$ is divisible by 5" for $a, \; b \in Z$. Prove that ρ is an equivalence relation on Z. Show that there are three distinct equivalence classes for ρ.

5.12.8 Examine if the relation ρ on the set Z is an equivalence relation

(a) $\rho = \{(a,b) \in Z \times Z : a.b \geq 0\}$

(b) $\rho = \{(a,b) \in Z \times Z : 3a + 4b \text{ is divisible by } 7\}$

(c) $\rho = \{(a,b) \in Z \times Z : |a - b| \leq 5\}$

(d) $\rho = \{(a,b) \in Z \times Z : a^2 + b^2 \text{ is a multiple of } 2\}$

5.12.9 Let S be the set of all positive divisor of 36. Define a relation ≤ on S by "$x \leq y$ only if x divides y" for $x, \; y \in S$. Prove that $(S, \leq)$ is a poset.

5.12.10 Let S be a finite set of three elements. How many different relations can be defined on S? How many of these are reflexive?

5.12.11 Let S be the set of all students in a coeducation college. Let ρ be a relation defined on S by "a ρ b if and only if a is the brother of b" for $a,b \in S$. Is ρ an equivalence relation? Is ρ a partial order relation? Justify.

5.12.12 Let ρ be a ternary relation defined on A^3 where $A = \{x \in N : 1 \leq x \leq 20\}$ by "$(a,b,c) \in \rho$ if and only if $a^2 + b^2 = c^2$" for $a,b,c \in A$.

(i) Find the values of $b,c \in A$ such that $(12,\ b,c) \in \rho$.

(ii) Find $(a,b,c) \in \rho$ such that $a,\ b,\ c$ are in arithmetic progression.

(iii) Is there any $a,b \in A$ such that $(a,b,4) \in \rho$?

(iv) Prove that $\exists\ m \in A$ such that $\left(a, 2m^2 + 4m,\ c\right) \in \rho$ for some $a,c \in A$.

5.13 Answer to the exercise 5.12

5.12.1 (i) (b) (ii) (b)

5.12.2 There are several possible examples, one possible set is as follows :

(i) $R = \{(2,2),(3,3)\}$

(ii) $R = \{(2,3),(3,2),(1,3)\}$

(iii) $R = \{(2,3)\}$

5.12.4 The equivalence relation

$$p = \{(1,1),(2,2),(3,3),(4,4),(5,5),(2,3),(3,2),(4,5),(5,4)\}$$

5.12.5

	Reflexive	Symmetric	Transitive
ρ_0	*F*	*F*	*F*
ρ_1	*F*	*F*	*T*
ρ_2	*F*	*T*	*F*
ρ_3	*F*	*T*	*T*
ρ_4	*T*	*F*	*F*
ρ_5	*T*	*F*	*T*
ρ_6	*T*	*T*	*F*
ρ_7	*T*	*T*	*T*

T means true. F means false.

5.12.7 $cl(0) = \{5n : n \in Z\}$

$cl(1) = \{5n \pm 1 : n \in Z\}$

$cl(2) = \{5n \pm 2 : n \in Z\}$

5.12.8 (a) No (b) Yes (c) No (d) Yes

5.12.10 2^9, 2^6

5.12.11 ρ is neither equivalence relation nor partial order relation

5.12.12

(i) $(b,c) \in \{(5,13),(9,15),(16,20)\}$

(ii) (3, 4, 5)

(iii) No

(iv) $m = 1$, $a = 8$, $c = 10$ so that $(8,\ 2.1^2 + 4.1,\ 10) \in \rho$

5.14 Summary

In this unit we have discussed various types of relations, reflexive, symmetric, transitive, antysymmetric, equivalence relation etc. elaborately. Relations are in general also a part of set thory. Relations are at its core no more than ordered pairs. Binary relations are common when studying database design through relational calculus and allow us to define things such an order.

References and Further Readings

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3. Robert R. Stoll, *Set Theory and Logic*, Dover Publications Inc., 1979.

4. P. R. Halmos, *Naive Set Theory*, springer, 1974.

5. E. Kamke, *Theory of Sets*, Dover Publishers, 1950.

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Sample Questions

Short answer type questions :

1. For each of the following decide whether it is a proposition or not and if it is, indicate whether it is true or false.

   (i) 16 is a prime number.

   (ii) $x^2 + 1 > 0$

   (iii) $\forall x \in R, x^2 \geq 0$ (R being the set of real numbers)

   (iv) Do your homework.

2. Let p, q, r denote the propositions

   p : The weather is nice.

   q : It is day-time.

   r : Madhab will wash his car.

   Write the following in symbolic form

   (i) It is night-time and Madhab will not wash his car.

   (ii) It is not the case that either it is day-time or the weather is nice.

   (iii) If it is day-time and the weather is nice then Madhab will wash his car.

   (iv) Either it is day-time and Madhab will wash the car or the weather is not nice.

3. Let p and q be any two logical statements and $r : p \rightarrow (\sim p \vee q)$. If r is a false statement then find the truth value of p and q.

4. Prove that the statement $\sim (p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$

5. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n \in N\}$ where N is the set of natural numbers then prove that $X \cup Y = Y$.

6. Let $A = \{1,2,3,4,5\}$. Determine the number of

   (i) Subsets containing three elements.

   (ii) Subsets containing the elements 1 and 2.

7. Let Z be the set of integers and a relation defined on the set Z by "$x \rho y$ if and only if $xy + y^2 = x^2 + 1$" for x, $y \in Z$.

   Give one counter example to show that ρ is not transitive.

8. Find all the equivalence relations on the set $A = \{1,2\}$.
9. Let p, q and r denote the propositions

   p : The light is on.

   q : The switch is down.

   r : The door is closed.

   Write the following propositions in English sentences.

   (i) $(p \wedge q) \leftrightarrow (\sim r)$

   (ii) $(p \vee \sim q) \rightarrow r$
10. If a and b are false statements and c and d are true statements then what are the truth values of the following propositions

    (i) $(\sim a) \wedge (c \vee \sim b)$

    (ii) $(c \wedge \sim d) \vee (a \vee \sim (\sim b))$
11. Let A and B be two sets containing 2 and 3 elements respectively. Find the number of subsets of $A \times B$ having 3 or more elements.
12. In a class there are 10 students with white shirts and 8 students with red shirts. 4 students have black shoes and white shirts. 3 students have black shoes and red shirts. There is no student with both white and red shirt. The number of students with white shirt or red shirt or black shoes is 21. Find the number of students who have black shoes.
13. Choose the correct option with justification :

    If s and r be two statements then the negation of the student $\sim s \vee (\sim r \wedge s)$ is equivalent to

    (i) $s \wedge \sim r$ (ii) $s \wedge (r \wedge \sim s)$ (iii) $s \vee (r \vee \sim s)$ (iv) $s \wedge r$
14. Find the converse and invese of the proposition $(p \wedge \sim q) \rightarrow r$.
15. Let $S = Z \times Z$ and ρ be defined on S by (a,b) ρ (c, d) if and only if $ad = bc$ $\forall (a,b),(c,d) \in S$. Show that ρ is not transitive.
16. Define the composition of two relations with an example.

Medium answer type questions :

17. Construct a truth table for the compound statement $\sim(p\vee\sim q)\rightarrow\sim p$ where $p,\ q,\ r$ denote primitive statements.
18. Let Z be the set of integers. Consider the predicates
    $p(x) : x < 1$ and $q(x) : x > 5$
    Determine which of the following statements are true and which are false.
    (i) $(\forall x \in Z)(p(x) \vee q(x))$
    (ii) $\{(\forall x \in Z)p(x)\} \vee \{(\forall x \in Z)q(x)\}$
    (iii) $(\exists x \in Z)(p(x) \wedge q(x))$
    (iv) $\{(\exists x \in Z)p(x)\} \wedge \{(\exists x \in Z)q(x)\}$
19. If $A,\ B,\ C$ be subsets of a universal set S, prove that
    $(A \cup B)\cap(B \cup C)\cap(C \cup A) = (A \cap B)\cup(B \cap C)\cup(C \cap A)$
20. Let ρ be a relation defined on C (the set of complex numbers) by
    "$(a + ib)$ ρ $(c + id)$ if and only if $a \le c$ and $b \le d$ for $a+ib, c+id \in C$.
    Show that ρ is a partial order relation.
21. Prove by using truth table that for any two primitive statements p and q,
    $p \rightarrow q \Leftrightarrow \sim p \vee q$
    hence show that for any three primitive statements p,q,r.
    $(r \rightarrow (p \rightarrow q)) \Leftrightarrow (p \wedge r) \rightarrow q$
22. Let p and q be two propositions as follows :
    p : If a mobile phone is good then it is not cheap.
    q : If a mobile phone is cheap then it is not good.
    Show that p is equivalent to q.
23. For any three subsets A,B,C of a universal set S show by using double inclusion method that
    $(B - A)\cup(C - A) = (B \cup C) - A$
24. Prove that the relation '$\le$' defined by $(a,b) \le (c,d) \Leftrightarrow ab \le cd$ for $a,\ b,\ c,\ d$ are integers ranging from 0 to 6, is a partial order relation.

Long answer type questions :

25. (a) Show by the truth tables that $p \wedge (q \vee r)$ and $(p \wedge q) \vee r$ are not equivalent but $(p \wedge (q \vee r)) \rightarrow (p \wedge q) \vee r$ is a tautology.

(b) Let $S = \{1, 2, 3, 4\}$ and $\rho = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(2,3),(3,2)\}$

Show that ρ is reflexive and symmetric but not transitive.

26. (a) If *A*, *B*, *C* are three subsets of a universal set *S*, prove that

$A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

(b) For the univese of all integers let *p*(*x*) and *r*(*x*) be the following open statements

$p(x)$: x is even.

$r(x)$: x is divisible by 5

Write the following two statements in symbolic form :

(i) If x is even then x is not divisible by 5.

(ii) There exists an even integer divisible by 5.

Determine whether each of the statements above is true or false. For each false statement provide counter example.

27. (a) Prove that the inverse of an equivalence relation is an equivalence relation.

(b) Using Venn diagram verify that for any three sets *A*, *B*, *C* of universal set *S*. $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

28. (a) Examine whether the relation ρ defined on the set $S = Z \times Z - \{(0,0)\}$ by (a,b) ρ (c,d) if and only if $ad = bc$ for $(a,b),(c,d) \in S$ is an equivalence relation.

(b) Verify that $[(p \leftrightarrow q) \wedge (q \leftrightarrow r) \wedge (r \leftrightarrow p)]$

$\Leftrightarrow [(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)]$ for primitive statement *p*, *q* and *r*.

29. (a) Show that for any two statements *p* and *q*, $p \rightarrow q$ is not equivalent to the converse $q \rightarrow p$ but equivalent to the contrapositive $\sim q \rightarrow \sim p$.

(b) For any three subsets *A*, *B*, *C* of a universal set *S* prove that

$A \times (B \setminus C) = (A \times B) \setminus (A \times C)$

30. (a) Let Z be the set of integers. Let $p(x), q(x), r(x)$ be the following open statements.

$p(x): x^2 - 11x + 30 = 0$

$q(x): x^2 + x - 12 = 0$

$r(x): x < 0$

Determine the truth value of the following statements with explanation.

$(\forall x \in Z)(p(x) \rightarrow \sim r(x))$

$(\forall x \in Z)(q(x) \rightarrow r(x))$

$(\exists x \in Z)(q(x) \rightarrow r(x))$

$(\exists x \in Z)(p(x) \rightarrow r(x))$

(b) Let A and B be two sets taken from a universal set S. Define quantified definition of $A \subseteq B$. From this definition prove that negation of $A \subseteq B$ (that is $A \not\subseteq B$)

$\Leftrightarrow (\exists\, x \in S)(x \in A \wedge x \notin B)$

31. Let A = {1,2,3,4,5,6,7}

(a) Find the number of proper subsets of A.

(b) How many subsets of A have odd cardinality?

(c) Find the number of subsets of A of 3 elements with least element 4.

(d) Find the number of subsets of A of three elements with least element is less than 5

(e) Find the number of subsets of A of three elements containing one odd integer and two even integers.

32. (a) Let ρ be an equivalence relation on a set S and $a,b \in S$. Prove that $cl(a) = cl(b) \Leftrightarrow a \rho b$.

Hence show that $cl(a)$ and cl(b) are either disjoint or equal.

(b) Find the equivalence classes determined by the relation ρ on Z defined by "a ρ b if and only if $a - b$ is divisible by 4" for $a,b \in Z$.

Answers to the Sample Questions

1. (i) & (iii) are statements out of which (i) is false and (iii) is true.
2. (i) $\sim q \wedge \sim r$ (ii) $\sim (q \vee p)$ (iii) $(q \wedge p) \to r$ (iv) $(q \wedge r) \vee \sim p$
3. p : True, q : False
6. (i) 10 (ii) 8
7. $0 \rho -1$ and $-1 \rho 2$ hold but $0 \rho 2$ does not hold.
8. $\rho_1 = \{(1,1),(2,2)\}$, $\rho_2 = \{(1,1),(2,2),(1,2),(2,1)\}$
9. (i) The light is on and the switch is down if and only if the door is not closed.
   (ii) If the light is on or the switch is not down then the door is closed.
10. (i) True (ii) False
11. $2^6 - {}^6C_0 - {}^6C_1 - {}^6C_2 = 42$
12. 10
13 (iv)
14. Converse : $r \to (p \wedge \sim q)$ Inverse : $(\sim p \vee q) \to \sim r$
17.

p	q	$p \vee \sim q$	$\sim (p \vee \sim q)$	$\sim (p \vee \sim q) \to \sim p$
T	T	T	F	T
T	F	T	F	T
F	T	F	T	T
F	F	T	F	T

18. (i) (ii) (iii) are false statements and (iv) is a true statement.
26. (b) (i) $(\forall x \in Z)\, p(x) \to \sim r(x)$

    (ii) $(\exists x \in Z)(p(x) \wedge r(x))$

    First statement (i) is false and the second statement (ii) is true. Counter example of first statement is $x = 10$.
30. (a) True, False, True, False

31. (a) 127 (b) 64 (c) 3 (d) 34 (e) 12

32. (b) $cl(0) = \{4n / n \in Z\}$

$cl(1) = \{4n + 1 / n \in Z\}$

$cl(2) = \{4n + 2 / n \in Z\}$

$cl(3) = \{4n + 3 / n \in Z\}$

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